EDUC 310/GRED 565

COURSE MATERIALS

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INTRODUCTION

This course materials booklet is written to provide elementary teacher candidates with the basic ideas of PreK-6 mathematics curriculum and the modern-day methods of teaching mathematics in those grades. The basic ideas include the development of the concepts of number; arithmetic of whole numbers; base-ten system; the emergence of fractions in the context of simple real-life activities requiring the extension of whole number arithmetic; decimals, percent, and ratio as alternative representations of fractions; geoboard geometry; probability as a blend of theory and experiments; and statistical data analysis. The methods of teaching the corresponding topics emphasize the power of visual representations of mathematical concepts, the use of physical materials (manipulatives) and support of digital technology. Manipulative tools include pattern blocks, square tiles, two-sided counters, linking cubes, geoboards, coins, and dice. Digital tools include electronic spreadsheets, computational knowledge engine *Wolfram Alpha*, and the Geometer's Sketchpad. The course materials are presented in the context of major modern-day national and state educational documents including Common Core State Standards Initiative (http://www.corestandards.org), Principles and Standard for School Mathematics by the National Council of Teachers of Mathematics (https://www.nctm.org), New York State Next Generation Standards (http://www.nysed.gov/next-generation-learning-standards), Standards for Preparing Teachers of Mathematics by the Association of Mathematics Teacher Educators (https://amte.net), and the Conference Board of the Mathematical Sciences (https://www.cbmsweb.org) recommendation for the preparation of mathematics teachers.

The course materials emphasize connections of mathematics to real life as many mathematical concepts and procedures are reflections on common sense and intuitive understanding, something that schoolchildren inherently possess. For example, when one cuts across the grass rather than walking along the pavement to get to the school building faster, one demonstrates intuitive understanding of the triangle inequality: the sum of any two side lengths of a triangle is greater than the third side length. Likewise, one can intuitively recognize that the distance from a point on the floor to the wall is measured along the perpendicular, and that a circle is a shape with the largest area given the length of its border. (Yet, it is not intuitively clear that square has the smallest perimeter among all rectangles with the given area).

Also, the course materials underscore the importance of elementary teacher candidates' deep understanding of mathematics. The need for deep understanding of the subject matter is at least two-fold. First, the modern-day students have been actively seeking answers to conceptual questions about procedural knowledge they gain through the study of school mathematics. Second, teachers must know the content of mathematics they are assigned to teach beyond their assigned grade level. The course materials consist of eleven chapters and Appendix which includes ten activity sets to be discussed in class, completed by students as part of the course work, and collected in a course portfolio to be used as the major instrument of assessment.

CHAPTER 1: TEACHING PREK-K MATHEMATICS

> "Representing numbers with various physical materials should be a major part of mathematics instruction in the elementary grades" (National Council of Teachers of Mathematics, 2000, p. 33).

1.1 Developing the concept of number

A number when presented as a symbol is an abstraction. For example, the number 3 is an abstraction. It becomes a concrete entity through the association with concrete objects, like three apples, three cookies, three cats, three birds, and so on. In that way, the number 3 is a common decontextualized characteristic of the four groups of the mentioned (and many more not mentioned) objects. Using different sets of objects with the same cardinality makes it possible to develop the concept of number. Historically, numbers were used to count objects and to record the results of counting. In order to do this correctly, counting skills have to be developed. The skills are based on the following mathematical knowledge, both conceptual and procedural. One has to know number names and the order of numbers. One has to use each number name only one time when counting and count each object only one time. One has to begin counting with the number one and know that the last name used in the process of counting is the total number of objects counted. Furthermore, the order in which objects are counted does not affect the total count and objects can be rearranged to facilitate making one-to-one correspondence between the objects and number names (counting labels). Research suggests that the basic principles of counting listed above have to be developed first in order to use counting as a skill. Although one can conceptualize the infinity of numbers (indeed, as long as we have a way of writing down integers as large as we want, every integer is followed by itself increased by one and, therefore, this process may never stop), in real life, the number of objects in a set is finite and thus the process of counting stops when the last object in the set is assigned a number name which is then called the cardinality of the set. That is, infinity, not having a clear frame of reference, belongs to much higher level of abstraction than an integer.

Conservation of a number is another skill that has to be developed. To conserve a number means to understand that the number of objects does not change when the objects are covered, put in several groups, or just rearranged. This understanding is especially important when counting objects loosely organized because the efficiency of counting depends on a way objects are arranged. For example, counting a number of seats at a round table or even at a rectangular table is more difficult than counting the same number of seats being lined up (Figure 1.1).

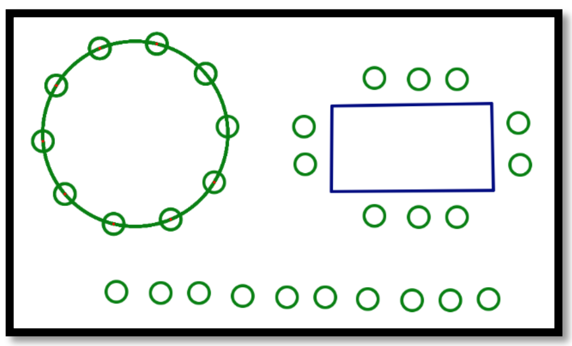

Figure 1.1. Practicing skills to conserve a number.

Furthermore, a skill of conserving a number can be applied in the case of distribution of any material like pouring water from a bottle into several glasses and realizing that the glasses contain the same amount of water that was in the bottle. The same is true when a cake is cut into several pieces and put on several plates. The original quantity of the cake does not change through this process. Subitizing is yet another concept associated with counting – it is the ability to correctly determine a small number of objects without counting them. Typically, human ability to subitize is limited to six objects.

The earliest distinction among numbers was based on the idea of even and odd numbers grounded in the experience of putting objects in pairs. In order to understand the idea, one has to be introduced to its genesis. History of mathematics tells us that the game of guessing odd or even with respect to the number of coins (or other small objects) held in hand was considered ancient even in the time of Plato (born in the 5^{th} century BC). The game, most likely, included pairing objects held in hand and, therefore, the concepts of even and odd numbers were associated with this action to decide the outcome of the game. Therefore, young children have to be introduced to this experience in order to develop conceptual understanding of even and odd numbers.

1.2 Learning to think with images

In ancient times, numbers and operations on numbers were associated with geometric images the creation of which was the result of an action. As Aristotle, student of Plato, put it, “the soul never thinks without an image” (cited in [Arnheim, 1969, p. 12]). For example, the image (created physically) shown in Figure 1.2 was associated with the summation of consecutive counting numbers 1, 1 + 2 , 1 + 2 + 3, 1 + 2 + 3 + 4, ... ; and the corresponding sums were (and are) called triangular numbers due to the image they form. Likewise, the image shown in Figure 1.3 was associated with the summation of consecutive odd numbers 1, 1 + 3 , 1 + 3 + 5, 1 + 3 + 5 + 7, ... , and those partial sums of odd numbers can be rearranged (through action) by using gnomons (Figure 1.4) to form squares, so that 1 + 3 = 4 , 1 + 3 + 5 = 9, 1 + 3 + 5 + 7 = 16, and 4 = 2 + 2, 9 = 3 + 3 + 3, 16 = 4 + 4 + 4 + 4.

Figures 1.5 and 1.6 show how the summation of consecutive even numbers can be reduced to either the summation of consecutive counting numbers or the summation of consecutive odd numbers. Indeed, Figure 1.5 shows that $2+4+6+8=2\cdot(1+2+3+4)$ and Figure 1.6 shows that $2+4+6+8=(1+3+5+7)+(1+1+1+1)=(1+3+5+7)+4$.

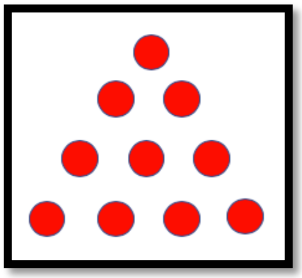

Figure 1.2. 1+2+3+4.

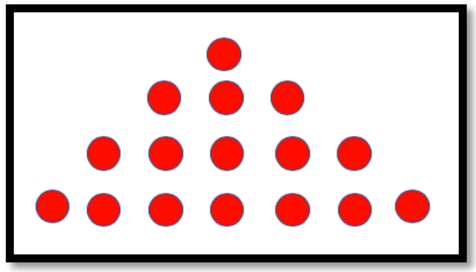

Figure 1.3. 1+3+5+7.

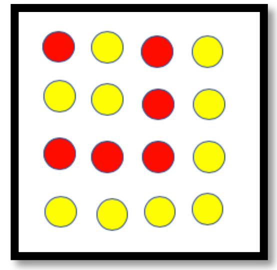

Figure 1.4. 1+3+5+7 = 4x4.

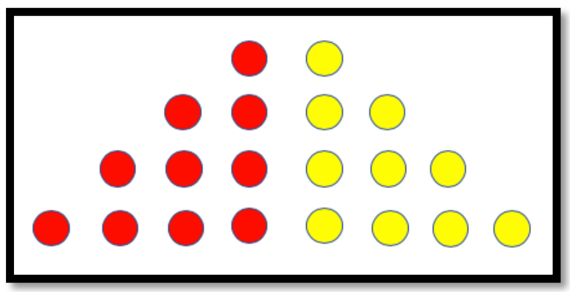

Figure 1.5. $2+4+6+8=2\cdot(1+2+3+4)$

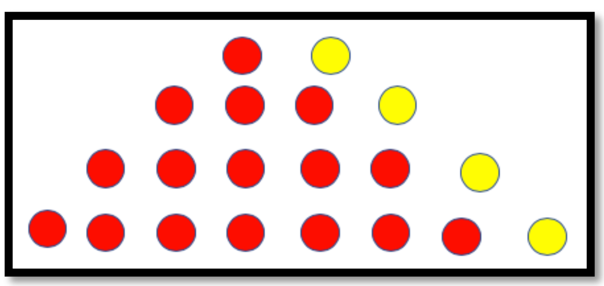

Figure 1.6. $2+4+6+8=(1+3+5+7)+4$

After the concept of number is developed, the next step is to explain that, in contextualized situations, numbers represent the results of counting or measuring. However, with young children's inability to conserve a number, often numbers have the spatial meaning rather than the numeric meaning. For example, by seeing on a picture three elephants and five mice (Figure 1.7) or two groups of five identical mice, each group taking different space on a picture (not shown here), a child may not see that five is greater than three or that the number of mice in each group in the same. In order to compare the numbers 5 and 3, they have to be presented through the same units, like it is shown in Figure 1.8. In order to see that the number of triangles is the same as the number of rectangles (Figure 1.9), one has to establish one-to-one correspondence between the two shapes. That is how one learns the numeric order of numbers visually, without counting. Large numbers are then compared through the operation of subtraction (or using base-ten system): 543 > 461 because 543 – 461 > 0 (or because there are more hundreds in 543 than in 461). Likewise, 543 > 537 because (with the same number of hundreds) 543 has more tens than 537. Finally, 543 > 542 because (with the same number of hundreds and tens) 543 has more ones than 542.

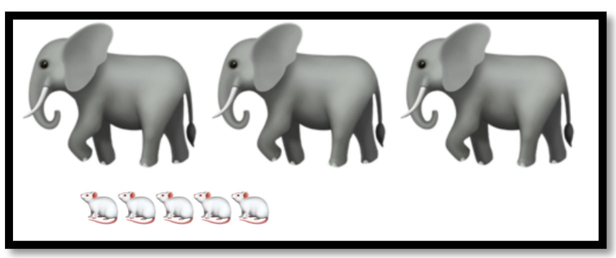

Figure 1.7. Is 3 greater than 5?

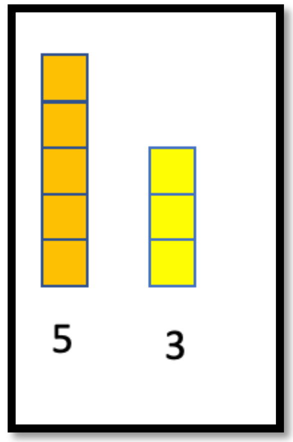

Figure 1.8. Comparing 5 to 3 using same units.

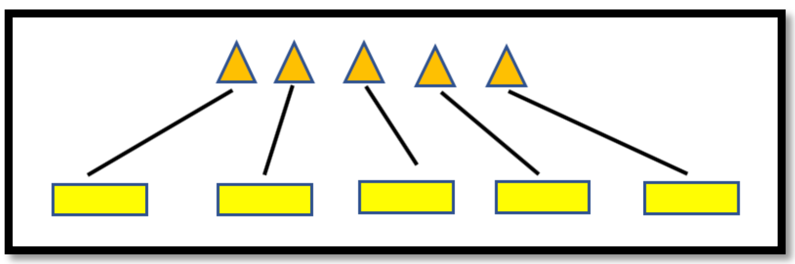

Figure 1.9. Comparing cardinalities through one-to-one correspondence.

1.3 Classification

The presence of the basic geometric shapes in the mathematics classroom (often called pattern blocks, see https://en.wikipedia.org/wiki/Pattern_Blocks) prompts the activity of their classification as a way of abstracting their common characteristics. Basic characteristics of geometric images (or pattern blocks) are shape, size, and color. Also, there may be the binary (yes – no) classification; e.g., triangles and not triangles, squares and not squares, and so on. Shapes can be put together to form other shapes. For example, one can make a rhombus, an isosceles trapezoid, and a (regular) hexagon out of two, three, and six equilateral triangles, respectively. Likewise, the hexagon can be made out of triangle, rhombus, and trapezoid. Such action-oriented relationships among pattern blocks are very important to be interpreted in terms of arithmetic for they lead to the concept of additive partition of a whole into unit fractions. Indeed, the latter relationship can be expressed through the equality $\frac{1}{6}+\frac{1}{3}+\frac{1}{2}=1$ which ascribes the unit fractions 1/6, 1/3, and 1/2 to triangle, rhombus, and trapezoid, respectively. Likewise, the triangle can be called 1/2 of the rhombus, 1/3 of the trapezoid, or 1/6 of the hexagon (Figure 1.10). That is, the same object can be assigned different fractional names depending in the unity of which the object is a part (alternatively, fraction).

Figure 1.10. Hexagon as a combination of other shapes.

1.4 Patterns and their symbolic description

Another major idea of PreK-K mathematics curriculum is creating, recognizing and describing patterns. Patterns and their symbolic description can be seen as early algebraic activities in the broad context of learning mathematics. The presence of a pattern implies repetition. For example, the pattern in which the pair triangle-square, is repeated as long as one wishes can be described as an AB-pattern where the letter A represents a triangle, and the letter B represents a square (Figure 1.11). The pattern in which the quadruple triangle-triangle-square-square is repeated as long as one wishes may be described as an AABB-pattern as well as an AB-pattern. In order to justify the latter description, one can introduce a pattern where hundred triangles are followed by hundred squares; that is, a set of two hundred shapes repeats itself over and over.

Describing such pattern in the form $\underbrace{AA...A}_{100\ times}\underbrace{BB...B}_{100\ times}$ cannot be accepted because this description can hardly be pronounced, and such a large number of letters cannot be accurately repeated. So, here one can learn about the efficiency of symbolic representation, one of the earliest introductions to formal reasoning in mathematics. In other words, recognizing that multiple visual patterns have the same symbolic description may be referred to as and early algebraic skill of generalization.

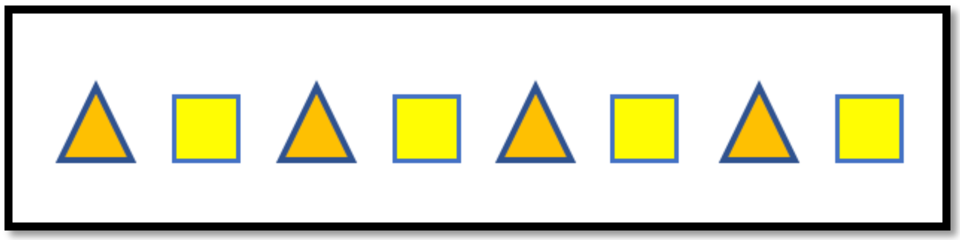

Figure 1.11. From visual to symbolic: an AB-pattern.

CHAPTER 2: TEACHING GRADES 1-4 MATHEMATICS

> "Teaching elementary mathematics requires both considerable mathematical knowledge and a wide range of pedagogical skills" (Conference Board of the Mathematical Sciences, 2001, p. 55).

2.1 From actions to operations

What do students know before they begin the study of arithmetical operations, the simplest of which is addition? They know such action-oriented notions as "one more than a number" and "two more than a number", something that can then be interpreted as augmenting a number by one and two, respectively. Such augmentation is called addition. Therefore, once the operation addition is formally introduced, 5 + 1 can be interpreted as 6 and 5 + 2 can be interpreted as 7. Immediately, the two arithmetical facts, 5 + 1 = 6 and 5 + 2 = 7, have to be committed to memory (including many similar addition facts). Moreover, students know doubles of some one-digit numbers. For example, doubles or 1, 2, 3, and 4 are 2, 4, 6, and 8. The knowledge of doubles can be connected to a skill of counting by twos (something that later can be interpreted as multiplying a number by two). Thus, the doubles can be interpreted as the sums of two equal numbers: 1 + 1 = 2, 2 + 2 = 4, 3 + 3 = 6, and 4 + 4 = 8. Furthermore, doubles represent even numbers[1].

In teaching operations, one has to begin with contextual problem solving when an operation is not needed to solve a problem because the numbers used by a context are small and the use of simple counting, physical manipulation and just common sense may be enough to get an answer.

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[1] A curious child may ask as to why this is true. Indeed, assuming that an even number represents the cardinality of a set of objects all of which can be put in pairs, a child might see a difference between the doubles of the numbers 3 and 4. In the spirit of early algebra, one can see 3 as a new unit (a single) which when doubles gives a pair. Likewise, one can see 4 as a new unit so that two units form a pair. Teachers have to keep in mind that what one considers being obvious, is not in fact obvious and, thereby, requires a sophisticated explanation,

For example, *if there are two cookies on Jonny's plate and three cookies on Annie's plate, how many cooking are on both plates*?

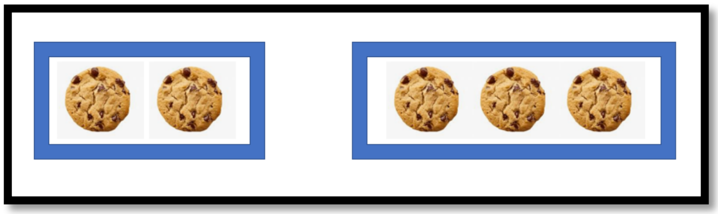

Figure 2.1. Counting vs adding.

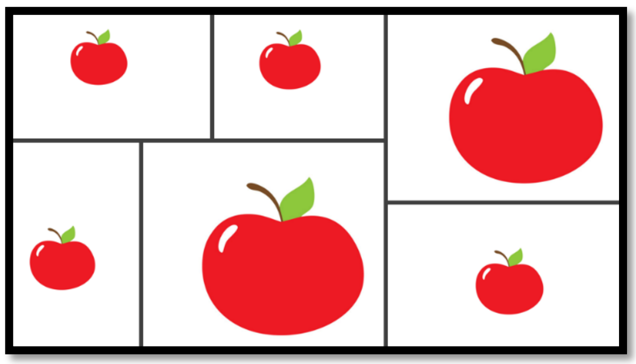

Figure 2.2. Counting vs subtracting.

A first grader can draw the pictures of two and three cookies (Figure 2.1) and find through counting that there are five cookies on both plates. Likewise (Figure 2.2), if out of six apples, four are small, one can count the remaining apples and conclude that there are two large apples. The latter problem may be characterized as a whole-part type subtraction problem (or take away when one inquires that if out of six apples two larger apples were eaten, how many apples remain). One can also introduce comparison context for subtraction: *if Jonny has 5 baseball cards and Ronnie has 9, how many more cards does Ronnie has than Jonny*? Another context is completion: *if Annie's homework incudes ten problems and she has solved six of them, how many problems does she need to solve in order to complete the homework*? All the problems can be easily solved, and one might wonder why we need to teach something called addition and subtraction. The answer deals with the need to solve similar contextual problems with large numbers. Young children are familiar with large numbers and know how to count beyond 100. However, finding the number of apples in two baskets with 78 and 183 apples or comparing the quantities of apples in the baskets through drawing and counting, are boring and time consuming activities. This boring aspect of mathematical problem solving that forces students to continue using strategies of drawing and counting, can serve as a motivation for the introduction of new concepts (operations) which are called addition and subtraction.

So, the problem with cookies can be solved by using the former operation and having the solution in the form 2 + 3 where each term is equal to the number of cookies on each plate. Likewise, the solution to the problem with 6 and 2 apples can be presented through the latter operation in the form 6 – 2. Here, the numbers 2 and 3 as well as 6 and 2 have situational referents in the form of cookies and apples, respectively. Knowing that 2 + 3 = 5 (through counting), one has to be encouraged to write the sum as 3 + 2 seeing it as two more than the number 3, and then find out that the "new" sum is also 5. In doing so, "students understand and use ... previously established results in constructing arguments" (Common Core State Standards, 2010, p. 6). Contextualization confirms that 2 + 3 = 3 + 2, that is, the order of plates through which one begins adding cookies does not change the sum. This is called commutative property of addition (the term may be omitted in the first grade classroom). Understanding this property, whereas not requiring knowledge of complicated terminology, enables one to recognize that knowing that, say, 25 is a number which is two more that 23 makes it possible to find the sum 2 + 23 by replacing it with the sum 23 + 2. At the same time, 6 – 2 is not equal to 2 – 6 as this operation, when contextualized, indicates impossibility of eating more apples than there are available.

2.2 Addition and subtraction in base-ten system

The next idea regarding addition and subtraction is the use of base-ten system. The reason for using base-ten system is purely biological – humans have ten fingers. Yet base ten is as difficult for young children as any other base different from base ten for elementary teacher candidates used to do arithmetic in base ten. Therefore, one should not take for granted that the addition fact $5 + 6 = 11$ is an obvious one. Intuitively, it is not obvious unless one understands that the meanings of the two ones in the right-hand side of the last equality are different. For that, one has to begin using place value charts and to learn doing addition using such charts (Figure 2.3). The use of the charts can be introduced as a game with the following rule: one may not have more than nine objects within the same section (place value) of the chart where addition process is carried out. If there are more than nine objects (i.e., ones) in the section for ones, then the double nine is the largest number that can be greater than nine after two numbers are added. Then one has to create ten ones, turn them into one ten and move it to the section for tens. Likewise, after adding two two-digit numbers, the largest number of tens in the section for tens is eighteen (ten of which make a hundred – a new place value). So, adding two-digit numbers requires a chart with three sections: for ones, for tens, and for hundreds. Subtraction can also be demonstrated using place value charts when subtraction is an action that can be described as taking away some quantity from a larger (or equal) quantity. But taking away on a place value chart (Figure 2.4) is another game with its own rules when ones are taken away until it is possible and what remains to be subtracted has to be engaged with tens (by breaking one ten in ten ones), then tens with hundreds, and so on. The Standards for Preparing Teachers of Mathematics (Association of Mathematics Teacher Educators, 2017) recommend using "the verb *regroup* rather than *borrow*" (p. 82, italics in the original) in the context of subtraction meaning, perhaps, that the verb borrow may trigger the idea of the need to return, something that, obviously, is not the case when the result of subtraction is not a negative number. At the same time, the word regrouping is used when one replaces ten ones with one ten (Figure 2.3).

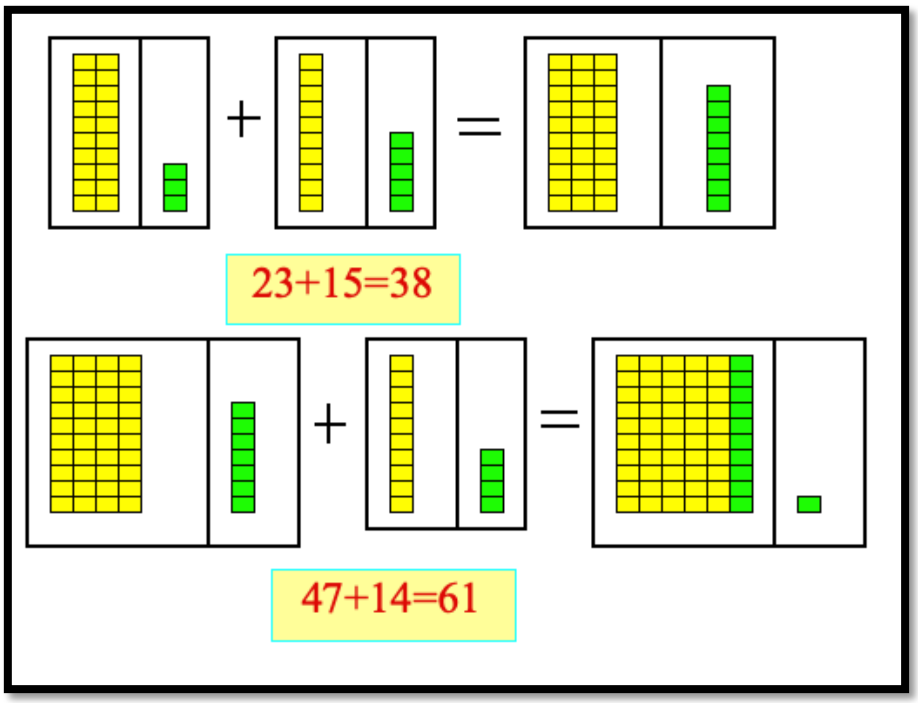

Figure 2.3. Addition without and with regrouping.

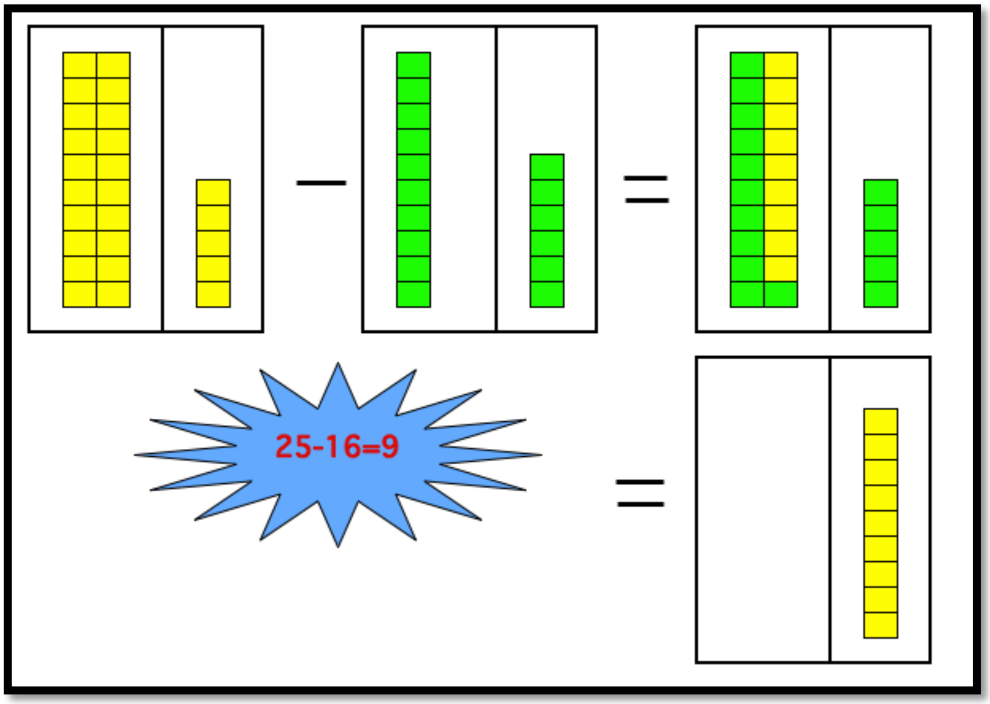

Figure 2.4. Subtraction as coloring.

2.3 Multiplication

If operations of addition and subtraction stemmed from the need to deal with contexts involving large numbers, multiplication stems from the need to avoid dealing with a large number of repetitions of the same, not necessarily large, number. For example, instead of counting by twos 100 times, starting from the number 2, that is, instead of carrying out addition in the sum

$\underbrace{2+2+...+2}_{100\ times}$, the sum can be presented through what is called the product, 100×2, of two numbers, 100 and 2, where the first number (factor) shows how many times the second number (factor) in repeated. However, just as in the case of addition (or subtraction), multiplication is introduced in the context of problem solving; yet, when the number of additive repetitions of a small number is also small. To this end, consider the following problem/question: *How many* *cookies ate in five boxes, each box having six cookies*? The number of cookies in five boxes (Figure 2.5) can be counted by writing down an addition sentence $6+6+6+6+6=30$.

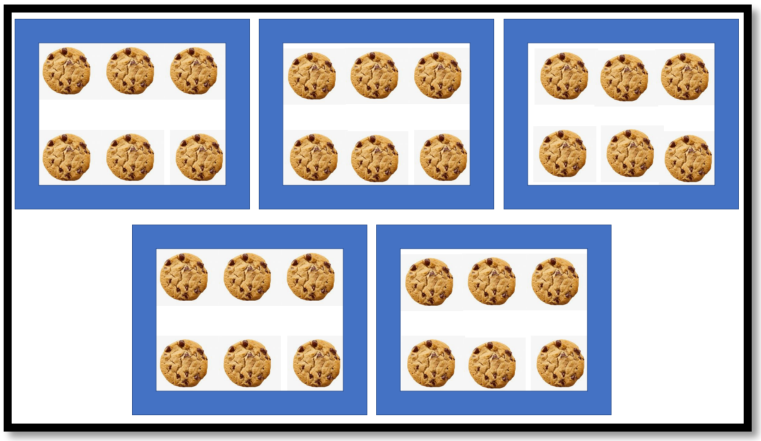

Figure 2.5. Five boxes with six cookies in each box.

Asking the class to read this sentence loudly and to say how many *times* the number six was repeated, yields the answer: *five times six equals thirty*. The last statement can then be written in the form $5\times 6=30$ in which a new symbol, "$\times$", appears. This symbol, which, as noted in (Association of Mathematics Teacher Educators, 2017, p. 82, is not to be confused with the letter x used as notation for a variable, can then be called the multiplication sign and the new operation connecting the numbers 5 and 6 is called multiplication. Many situations can be described through multiplication: counting the number of legs among five birds ($5\times 2=10$), the number of wheels on three cars ($3\times 4=12$), or the number of sides among four pentagons ($4\times 5=20$). Note that while all the three cases do not require the use of multiplication, a large number of birds, cars, or pentagons would make repeated addition a time consuming process requiring one to carry out repeated addition of a large number of addends. As stated in the Standards for Preparing Teachers of Mathematics (Association of Mathematics Teacher Educators, 2017), "well-prepared beginning Pre-K to Grade 2 mathematics teachers recognize the relationship between the content that precedes addition and subtraction (e.g., counting and cardinality) and the content that follows (e.g., multiplication and division)" (p. 51).

The next step towards developing multiplication skills is to show that multiplication is commutative, that is, $5\times 6=6\times 5; 5\times 2=2\times 5; 3\times 4=4\times 3; 4\times 5=5\times 4$. To this end, one can use a geometric representation of multiplication as a rectangular array comprised of unit squares (square tiles in the context of using manipulatives for a visual and tactile representation) as shown in Figure 2.6. One can see that the number of unit squares within the rectangle can be counted both as 3×6 and 6×3. This shows how a more sophisticated operation, multiplication, requires more complicated demonstration of commutativity than addition (i.e., geometrization rather than counting). Also, this is an example when one can generalize from a single instance; that is, seeing that $3\times 6=6\times 3$ implies $a\times b=b\times a$. Note that the equality $5\times 6=6\times 5$ can be justified by taking out a cookie from each of the five boxes and creating a new box with five cookies so that the number of cookies in six boxes can be written as 6×5. However, such demonstration does not allow for a convincing generalization from a single example.

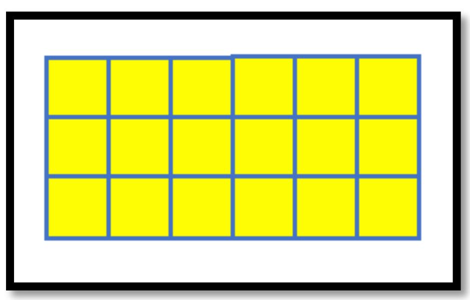

Figure 2.6. Multiplication is commutative.

The next activity deals with the multiplication table – a collection of multiplication facts for small numbers. Consider the case of the multiplication table of size ten (Figure 2.7). It is educationally beneficial to have students fill the table with the products of two integers by using conceptual understanding of multiplication as repeated addition.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Figure 2.7. The multiplication table created by *Wolfram Alpha*.

The first row of the table develops from the number 1 through counting by ones. The second row of the table develops from the number 2 through counting by twos. The third row of the table develops from the number 3 through counting by threes. The fourth row of the table can be developed either through counting by fours beginning from the number 4 or by doubling the numbers in the second row. The doubling phenomenon is demonstrated in Figure 2.8 in the case of $24 = 6 \cdot 4 = 2 \cdot (6 \cdot 2) = 2 \cdot 12$. The fifth row of the table develops from the number 5 through counting by fives. The sixth row of the table can be developed either through counting by sixes beginning from the number 6 or by doubling the numbers in the third row. The doubling phenomenon is demonstrated in Figure 2.9 in the case of $24 = 4 \cdot 6 = 2 \cdot (4 \cdot 3) = 2 \cdot 12$. The seventh row of the table develops from the number 7 through counting by sevens. The eighth row of the table can be developed either through counting by eights beginning from the number 8 or by doubling the numbers in the fourth row. The ninth row of the table can be developed either through counting by nines beginning from the number 9 or by tripling the numbers in the third row. Alternatively, finger multiplication can be introduced. For example, in order to find the product $6 \cdot 9$, one bends the sixth finger counting from the left on both hands and reads the first and the last digit of the product as, respectively, the number of fingers to the left and to the right of the bended finger. This rule suggests that the sum of digits in the product of any integer not greater than 10 multiplied by 9 is always equal to 9 (as out of ten fingers, one finger is bended). For example, this observation can be confirmed numerically as $6 \cdot 9 = 54$ and 5 + 4 = 9. Finally, the tenth row of the table develops from the number 10 through counting by tens.

The multiplication table offers many opportunities for conceptual explorations using hands-on activities. Consider the table shown in Figure 2.7. If such table is provided in hard copy with the main diagonal (top/left – bottom right) displayed, one can be asked to fold the paper table along this diagonal with numbers looking face-up and then punch a hole at any number on one side to see which number is affected on the other side. In Figure 2.7 two tens are highlighted in the table. The ten on the right-hand side represents 5x2, the ten on the left-hand side represents 2x5. The same can be said about the number 24. In that way, a commutative property of multiplication can be revisited through a hands-on activity.

Also, one can be asked to add numbers equidistant from the borders of the table within a column (or row). As shown in Figure 2.10, the sums 3 + 30, 6 + 27, 9 + 24, and so on are the same. How can this be explained? In terms of multiplication, we have

$$3 = 1 \cdot 3, 30 = 10 \cdot 3, 3 + 30 = 1 \cdot 3 + 10 \cdot 3 = 3 \cdot (1 + 10) = 3 \cdot 11 = 33,$$

$$6 = 2 \cdot 3, 27 = 9 \cdot 3, 6 + 27 = 2 \cdot 3 + 9 \cdot 3 = 3 \cdot (2 + 9) = 3 \cdot 11 = 33,$$

$$9 = 3 \cdot 3, 24 = 8 \cdot 3, 9 + 24 = 3 \cdot 3 + 8 \cdot 3 = 3 \cdot (3 + 8) = 3 \cdot 11 = 33.$$

In terms of action, this phenomenon can be explained using images of the entries in the multiplication table in the form of rectangular arrays. Figure 2.11 shows how moving a set of three counters from a larger array of counters exemplifies the replacement of a pair of numbers equidistant from the top and the bottom border of the table by another such pair so that the total number of counters in both arrays stay the same. One can note that the sums are not only all the same but are multiples of the number 11 which is one greater that the size of the table.

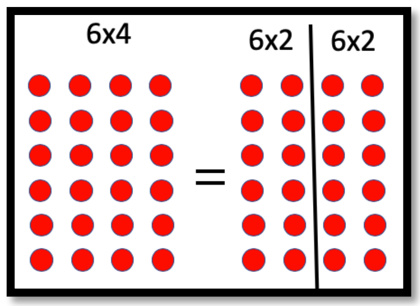

Figure 2.8. Doubling phenomena in the multiplication table.

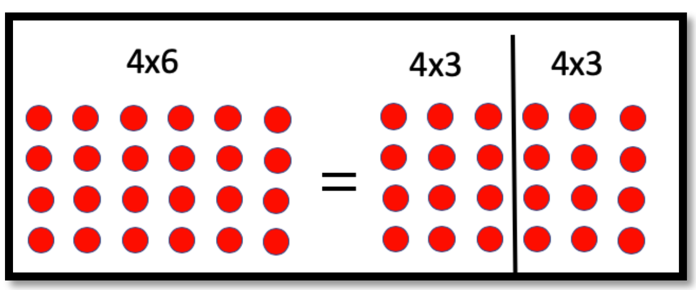

Figure 2.9. Doubling phenomena in the multiplication table.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Figure 2.10. Exploring patterns in the multiplication table.

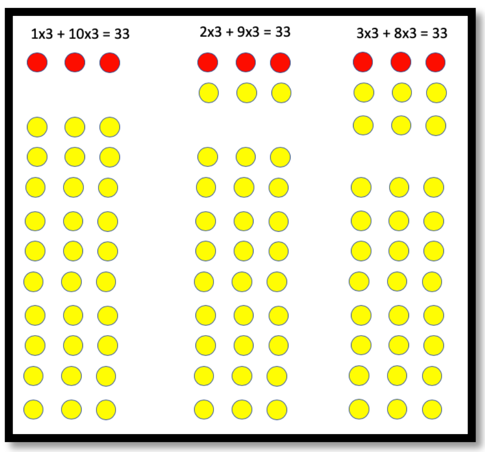

Figure 2.11. The sums of products equidistant from top and bottom are the same.

Consider the number 50 in the last row of the table of Figure 2.7. If, starting from this number, one moves up-right diagonally, the numbers on such a path are 54, 56, 56, 54, and 50. One can see a symmetrical pattern when the path starts with 50 and ends with 50 by going up and down. In order to explain this pattern, consider the arrays shown in Figure 2.12. The far-left array represents the number $50 = 5 \cdot 10$. The next array represents the number $54 = 6 \cdot 9$; that is, out of the array $5 \cdot 10$ one has to create the array $6 \cdot 9$. To this end, the 10th row with five circles is used in order to complement five rows of five circles with a circle each, and one needs another four circles to have a $6 \cdot 9$ array. This explains that the difference between 54 and 50 is 4. In order to make a transition from the array $6 \cdot 9$ to the array $7 \cdot 8$, two more circles are needed. Then the array $7 \cdot 8$ is replaced by array $8 \cdot 7$ and this explains the presence of two consecutive 56's on the path in question. Then from the array $8 \cdot 7$ one moves to the array $9 \cdot 6$ and then to array $10 \cdot 5$.

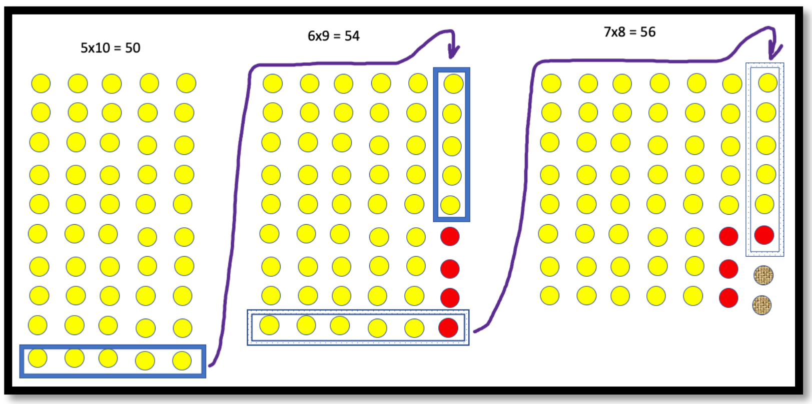

Figure 2.12. Moving diagonally up from bottom-left.

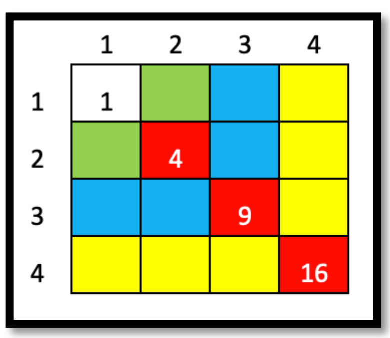

Figure 2.13. Moving along the main diagonal by adding a double plus one.

Another activity deals with the sequence of perfect squares that the multiplication table contains. To this end, one can write down the numbers residing in the main diagonal of the multiplication table: 1, 4, 9, 16, and note that the differences between two consecutive numbers in this sequence are consecutive odd numbers. This observation can be explained by noting that in order to build a square of size $n + 1$ from the square of size n, one has to cover two adjacent sides with n unit squares and then add one unit square (Figure 2.13, cases $n = 1, 2, 3$). In a different form, this phenomenon can be seen in the Figures 1.3 and 1.4. The multiplication table provides many opportunities for elementary teachers to "study the mathematics they will teach in depth ... [rather than] to rely on their past experiences as learners of mathematics" (Conference Board of the Mathematics Sciences, 2012, p. 23). For more activities with the multiplication table see Abramovich (2007).

2.4 Division

Why do we need to teach division as a new arithmetical operation? To answer this question, one can say that there are problems with a missing factor when the known numbers involved are large and such problems require a special mathematical operation called division. For example, *how can one put 781 apples into 11 boxes evenly*? This requires finding a number which when repeated 11 times gives 781. That is, to solve the equation $11 \cdot x = 781$ Alternatively, due to the commutative property of multiplication the last equation can be re-written as $x \cdot 11 = 781$ to mean that one has to find the number of repetitions of 11 needed to reach 781 (or the number which smaller than 781 by a number less than 11). But in order to develop a skill in carrying out the described processes, the teaching of division must begin with problems involving small numbers that can be solved through a hands-on approach. To this end, two problems can be considered.

Problem 1. *Billy has 12 apples. He wants to give ALL the apples to his friends, Alan, Bob, and Tom, in a fair way. How many apples would each friend get?*

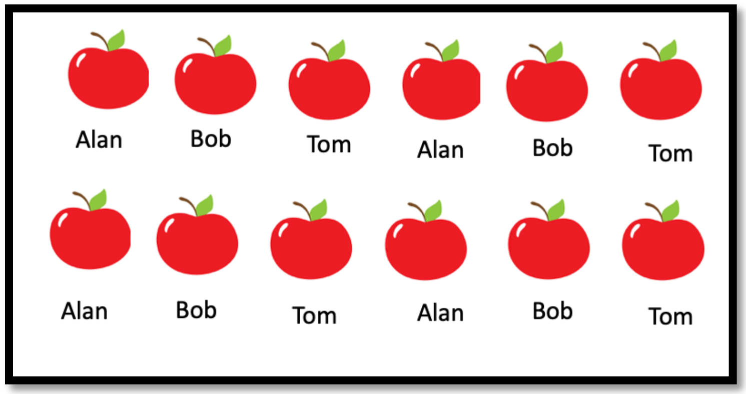

Figure 2.14. Dividing 12 apples among three people.

Problem 2. *Billy has 12 cookies. He wants to make servings, 3 cookies in each serving. How many servings can he make?*

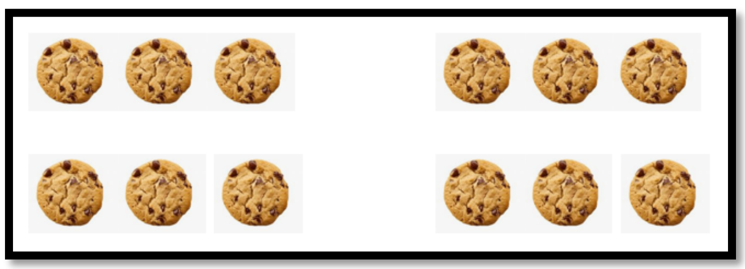

Figure 2.15. Measuring 12 cookies by three cookies as a serving.

One can see (Figure 2.14) how 12 apples are gradually partitioned among three people so that each person received four apples. In terms of multiplication, Figure 2.14 represents the multiplication fact $3 \cdot 4 = 12$ where the second factor (the number of apples given to each of the three individuals) was not originally known and was found through a hands-on activity.

Likewise, one can see (Figure 2.15) how 12 cookies were used to create four serving of three cookies in each. In terms of multiplication, Figure 2.15 represents the multiplication fact

$4 \cdot 3 = 12$ where the first factor was not originally known and was found through a hands-on activity.

Didactically speaking, Problem 1 introduces the partition model for division. Problem 2 introduces the measurement problem for division. Both terms reflect actions that were used to solve the problems.

In order to understand that division is not a commutative operation, one has to compare $12 \div 3$ to $3 \div 12$ by putting the two operations in contexts of Problem 1 and Problem 2. One cannot measure 3 cookies by 12 cookies (Problem 2) and cannot divide 3 apples among 12 people (Problem 1) ... unless apples are cut in equal pieces, something that motivates the introduction of fractions to serve contexts where integers do not work. Even allowing for apples to be cut into equal pieces, dividing 12 apples among 3 people gives a different quantitative result than dividing 3 apples among 12 people and, in that sense, such comparison demonstrates the absence of commutativity for division. Using the context of Problem 2, one can show that whereas division by zero is not possible, the operation leads to the concept of infinity (already discussed above in section 1.1, Chapter 1). Indeed, out of 12 (or any number of) cookies one can create as many empty servings, as one wishes.

The multiplication table can be used to solve both partition and measurement division problem for relatively small numbers. For example, in order to find the result of division of 56 by 7, one can use the table and find a number which when multiplied by 7 (i.e., repeated seven times) yields 56. However, beyond the multiplication table shown in Figure 2.7, like in the case of $781 \div 11$, one needs to use a special tool known as the algorithm of long division (which, as will be shown in Chapter 8, does not just replace a calculator but, better still, the algorithm provides a number of conceptual clarifications in the context of fractions). In order to explain how the long division works, one has to demonstrate that each step of the algorithm can be put in a real-life context of division which requires common sense only; in other words, to make sure that a student is able "to justify, in a way appropriate to the student's mathematical maturity ... where a mathematical rule comes from" (Common Core State Standards, 2010, p. 4). To this end, consider the following situation: *A school district received 462 oranges which have to be divided evenly among three cafeterias located in elementary, middle, and high schools*. The oranges arrived in boxes as follows (Figure 2.16): there were four large boxes with the label 100, six small boxes with the label 10 and two individually wrapped up oranges. In order to resolve the situation, a manager, without any knowledge of mathematics, set aside three large boxes (each of which to be sent to the corresponding school) and only after that unpacked the fourth box which included ten small boxes with the label 10 on each. The next step was to deal with 16 small boxes and put them into three places. This resulted in five small boxes to be sent to each of the three schools. Finally, the remaining box was unpacked, and 12 oranges were evenly divided among the three schools. As a result, each cafeteria received 154 oranges.

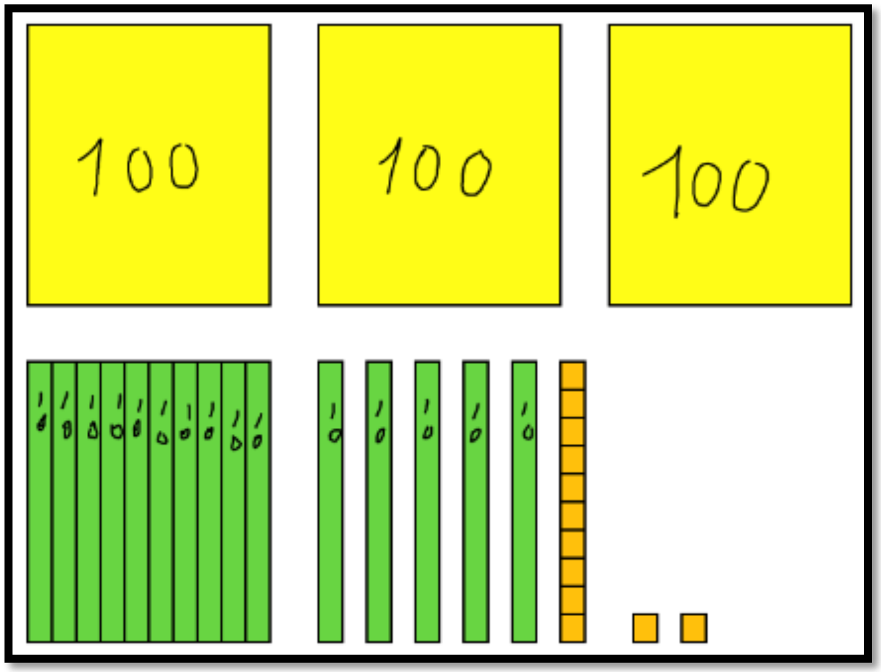

Figure 2.16. Dividing 462 oranges among three cafeterias evenly.

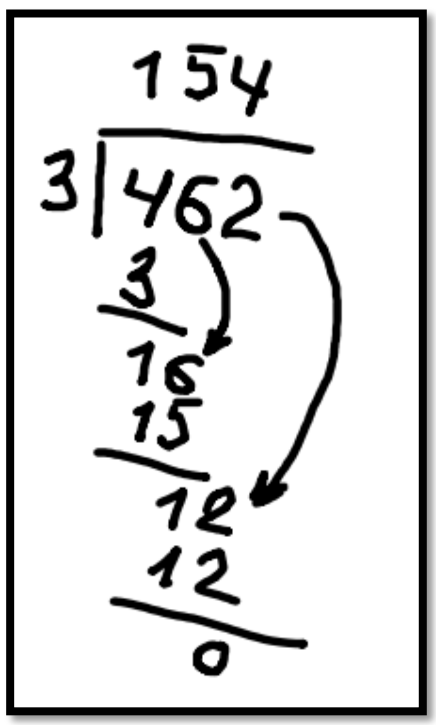

Figure 2.17. From context (manager's task) to decontextualization (long division).

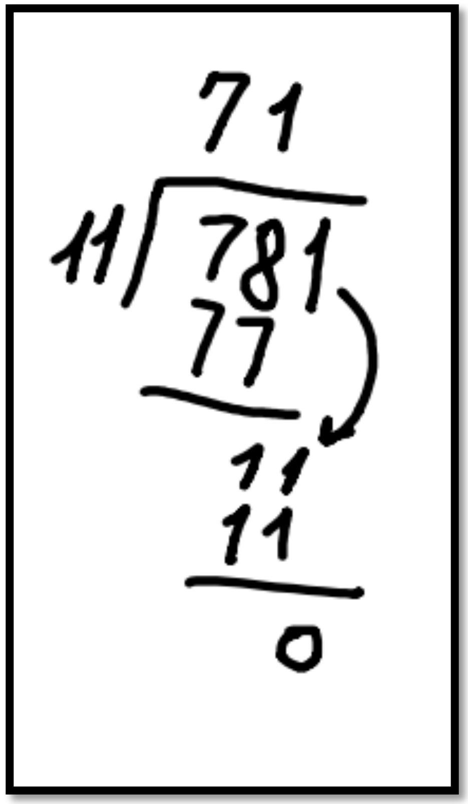

Figure 2.18. A problem with oranges solved through the algorithm of long division.

The contextual description of the solution of the problem with sorting oranges can be presented in the decontextualized form shown in Figure 2.17, in which 3 is divided into 465 by first dividing 3 into 4 (large boxes). The result, one box (for each school), becomes the first digit of the quotient. The next step is to subtract 3 (large boxes) from 4 (large boxes) to have 1 (large box). Now, one has to unpack this remaining box which reveals 10 small boxes and bringing down 6 exemplifies having the total of 16 small boxes which are to be evenly divided among three schools. Dividing 3 into 16 (small boxes) results in the number 5 – the second digit of the quotient (contextually, five small boxes having been sent to each school along with a large box). Subtracting 15 (small boxes) from 16 (small boxes) yields a single small box which, when unpacked, reveals 10 individual oranges, thus leaving to the manager to partition 12 oranges among three schools.

That is, dividing 3 into 12 yields the number 4 – the last digit of the quotient. This completes the process of division without producing a remainder. Numerically, the result can be written in two alternative forms: $462 \div 3 = 154$ or $3 \cdot 154 = 462$. This is an example of what it means to "bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize* ... and the ability to *contextualize*" (Common Core State Standards, 2010, p. 6).

Finally, the original problem of putting 781 oranges into 11 boxes evenly can be decontextualized allowing for the direct application of the algorithm of long division (Figure 2.18). One can note that in both cases, division did not produce a remainder. Furthermore, an important observation is that remainder may not be greater than or equal to divisor. This observation would play an important role in the case of developing decimal representations of common fractions.

CHAPTER 3: CONCEPTUAL SHORTCUTS

"Well-prepared beginners value varied approaches to solving a problem" (Association of Mathematics Teacher Educators, 2017, p. 10).

3.1 Conceptual shortcuts in arithmetic calculations

Doing mathematics is rooted in using the blend of computations and argumentation. Even in the age of computers, mathematical reasoning is often more preferable strategy of doing mathematics than computing. Conceptual shortcuts in mathematics education can be defined as problem-solving strategies based on insight that makes solution of a problem less computationally involved. Yet, insight is difficult to formalize as a computational algorithm. Consider the case of the combination of addition and subtraction with three numbers involved: $26 + 9 - 7$. Without much thinking, yet following the order of operations as presented, one can first do $26 + 9$, something that requires the use of addition with regrouping by decomposing 9 into 4 and 5, so that $26 + 9 = 26 + 4 + 5 = 30 + 5 = 35$. The next step is to subtract 7 from 35 which requires one to 'borrow' two from one of the three tens included into 35 as follows: $35 - 7 = 35 - 5 - 2 = 30 - 2 = 20 + (10 - 2) = 20 + 8 = 28$. However, a conceptual shortcut is to begin with subtracting 7 from 9 to get 2 and then add 2 to 26 to get 28. Of course, not all cases of addition and subtraction allow for the use of a conceptual shortcut of that kind by a young learner of mathematics. For example, in the 2$^{\text{nd}}$ grade, $26 + 5 - 9$ cannot be computed without decomposing and borrowing/regrouping. So, one would have to either add 26 and 5 or subtract 9 from 26; both operations require knowledge of arithmetic in base ten, something that was avoided in the former example. Note that young students do not yet have knowledge to replace $5 - 9$ by $-(9 - 5)$, so that $26 + 5 - 9 = 26 - (9 - 5)$

= 26 – 4 = 22, something that can be seen as a conceptual shortcut also, but requiring mathematical skills of the 6th grade.

3.2 Conceptual shortcuts in multiplying two-digit numbers

Another example of a conceptual shortcut could be in multiplying 54 by 12 using a geometric representation of the product as shown in Figure 3.1. The product can be represented by the sum of areas of four rectangles as follows (assuming that students know formula for area of a rectangle):

$$54 \cdot 12 = 50 \cdot 10 + 50 \cdot 2 + 4 \cdot 10 + 4 \cdot 2 = 500 + 100 + 40 + 8 = 648 .$$

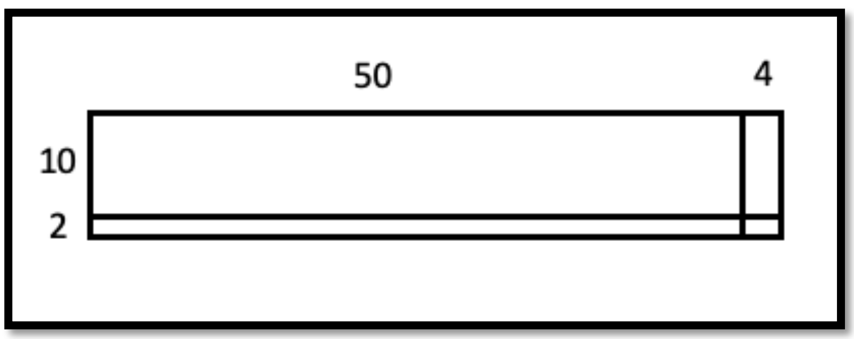

Figure 3.1. Distributive property of multiplication over addition: $(50+4)\cdot(10+2)$.

There are several ways to multiply numbers without using a calculator based on the distributive property of multiplication over addition and the visual method is one of them. As mentioned by the Conference Board of the Mathematical Sciences (2012) – an umbrella organization comprised of 16 professional societies concerned, in particular, with mathematics teacher education, teacher preparation programs should help teachers “to develop mathematical habits of mind ... [by using] mathematical drawings, diagrams, manipulative materials, and other tools to illuminate, discuss, and explain mathematical ideas and procedures” (p. 33).

3.3 Conceptual shortcuts in the summation of integers

Conceptual shortcuts can be used in the context of summation of a large number of consecutive integers (or those in arithmetic progression when the difference between any two neighboring integers is the same; e.g., consecutive odd numbers 1, 3, 5, 7, ... are in arithmetic progression with the difference two). The following example is well known from the history of mathematics, the importance of knowing which, along with the social contexts "affect [mathematics] teaching and learning" (Association of Mathematics Teacher Educators, 2017, p. 107).

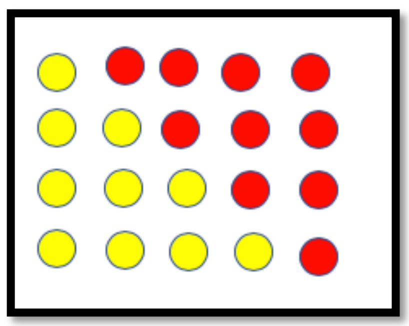

Figure 3.2. An image of $1+2+3+4=\frac{4\cdot(4+1)}{2}$.

Once upon a time, a teacher asked his young students to find the sum 1 + 2 + 3 + ... + 100. Among the students was Carl Friedrich Gauss (1777-1855), who is now considered the greatest mathematician of all time. Young Gauss, as the legend goes, noted that if the sum is written backwards and added to the sum with the terms written forward, then the terms equidistant from the beginning and the end of the sums when added together have the same sum, 101. Indeed, 1 + 100 = 2 + 99 = 3 + 98 = ... = 50 + 51 = 101. There are hundred such pairs if the sum is computed twice. Therefore, the result has to be divided by two. In that way, 1 + 2 + 3 +…+ 100 = $\frac{101\cdot 100}{2}=5050$. In general, $1+2+3+...+n=\frac{n\cdot(n+1)}{2}$, as

$$\begin{aligned}&(1+2+3+...+k+...+n)+[n+(n-1)+(n-2)+...+(n-k+1)+...+1]\\&=(1+n)+[2+(n-1)]+[3+(n-2)]+...+[k+(n-k+1)]+...+(n+1)=n(n+1).\end{aligned}$$

The case n = 4 is shown in Figure 3.2.

3.4 Conceptual shortcuts in solving word problems

Conceptual shortcuts can be used to solve word problems without recourse to algebraic computations. Consider the following problem: *If Jim spent \$20 at a book sale to buy at least one of each of the books priced \$5 and \$2, how many books of each price did he buy?* To solve the problem, note that if Jim buys one, two, three or four books priced \$5, then the cost will be \$5, \$10, \$15, or \$20, respectively, and what would be left for buying \$2 books are \$15, \$10, \$5, or nothing. Only \$10 can be spent on \$2 books to buy five of them. Thus, Jim can only buy two books priced \$5 and five books priced \$2. This solution is shown in the diagram of Figure 3.3.

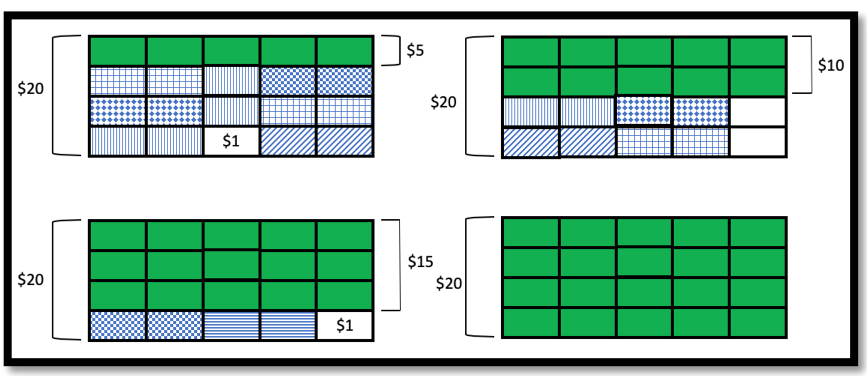

Figure 3.3. Solution to the book sale problem.

Whereas one may not see the use of a conceptual shortcut here, at a formal level, when students in Grade 6 “extend previous understanding of arithmetic to algebraic expressions” (Common Core State Standards, 2010, p. 43), the problem can be described through the equation $5x + 2y = 20$, where the unknowns $x \neq 0$ and $y \neq 0$ stand for the number of books priced \$5 and \$2, respectively, that Jim bought. One can see that both $5x$ and 20 are divisible by 5 and therefore, $2y$ should be divisible by 5. This implies that y may only be equal to 5 (as already $y = 10$ implies $x = 0$, thus contradicting the assumption $x \neq 0$).

These are examples of problems that can be solved without (grade-related) complexity of computation through a recourse to argument followed by a simple, perhaps purely mental, computation. That is, how mathematical knowledge develops through a combination of argument and computation. Both argument and computation may be free from the use of paper and pencil. Sometimes, argument can be supported by an image, so that “students can ... draw diagrams of important features and relationships” (*ibid*, p. 6).

CHAPTER 4: DECOMPOSITION OF INTEGERS INTO SUMMANDS

> "Elementary teachers should know ways to use mathematical drawings ... to illuminate, discuss, and explain mathematical ideas and procedures" (Conference Board of the Mathematical Sciences, 2012, p. 33).

Additive decomposition of integers can be put in context in order to begin with action and then describe this action through numbers using the "**w**e **w**rite **w**hat **w**e **s**ee" (W^4S) principle (Abramovich, 2017). For example, in order to build two towers out of ten square tiles and arrange them from the smallest to the greatest, one has to develop the following sets of two towers (Figure 4.1) and then describe the sets as follows: $10 = 1 + 9 = 2 + 8 = 3 + 7 = 4 + 6 = 5 + 5$.

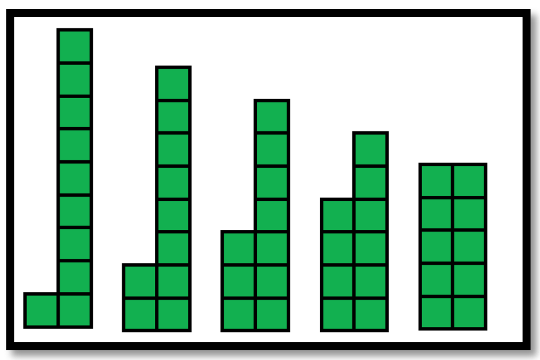

Figure 4.1. Additive decomposition of 10 in two integers.

A similar strategy, yet not immediately clear how to use it, is to build three towers out of ten square tiles and arrange them from the smallest to the greatest. Such towers are shown in Figure 4.2 and the strategy used here is to begin with the highest possible tower and then gradually decrease its size by adding each new tile to the smallest towers keeping the condition of their arrangement from the smallest to the greatest in place. Now, the diagram of Figure 4.2 enables one to decompose 10 into three addends arranged from the least to the greatest:

$$10 = 1 + 1 + 8 = 1 + 2 + 7 = 1 + 3 + 6 = 2 + 2 + 6 = 1 + 4 + 5 = 2 + 3 + 5 = 2 + 4 + 4 = 3 + 3 + 4.$$

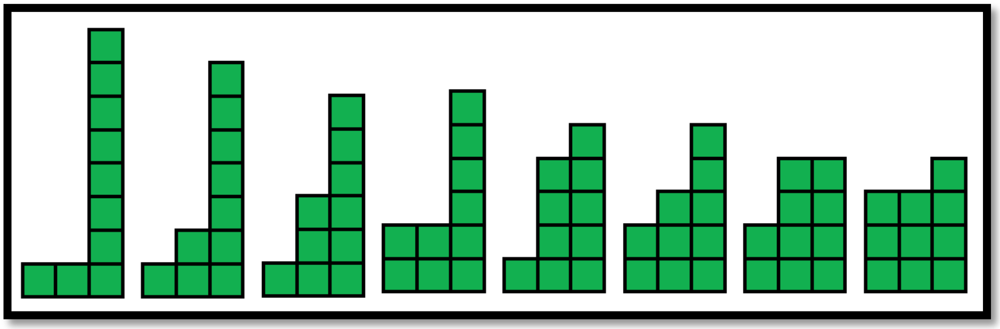

Figure 4.2. Additive decomposition of 10 in three integers (focus on the largest tower).

Another strategy is to focus on the smallest tower and decompose the remaining blocks in two addends how it was done in the case of Figure 4.1. Thus we have (Figure 4.3) four sets of three towers with one tile used for the first tower: $10 = 1+1+8 = 1+2+7 = 1+3+6 = 1+4+5$; three sets of three towers with two tiles used as the first tower: $10 = 2+2+6 = 2+3+5 = 2+4+4$; and a single set with three tiles used as the first tower: $10 = 3+3+4$

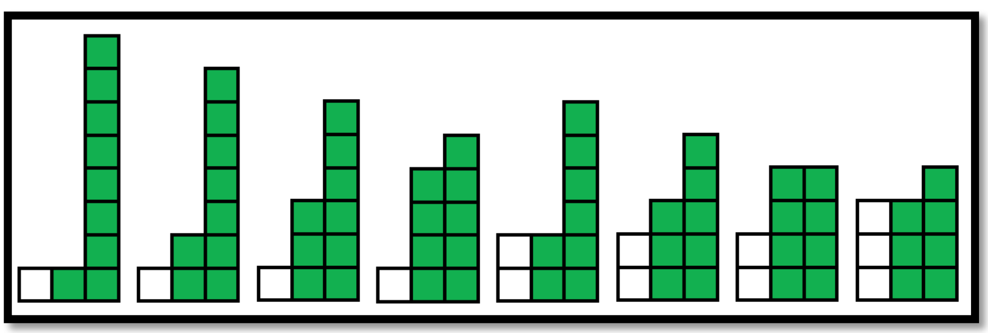

Figure 4.3. Additive decomposition of 10 in three integers (focus on the smallest tower).

Figure 4.4 shows decomposition of 10 into four addends arranged from the least to the greatest focusing on the largest (the fourth) tower:

$$10 = 1+1+1+7 = 1+1+2+6 = 1+1+3+5 = 1+2+2+5 = 1+3+3+3$$
$$= 1+1+4+4 = 1+2+3+4 = 2+2+2+4 = 2+2+3+3.$$

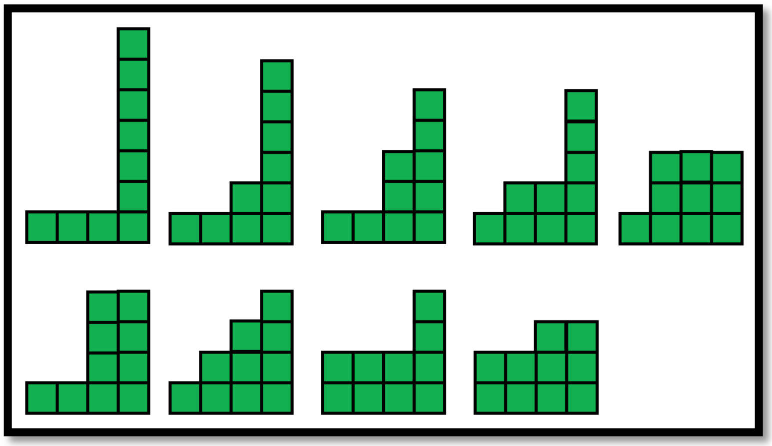

Figure 4.4. Additive decomposition of 10 in four integers (focus on the largest tower).

Remark 1. If the requirement for Figure 4.1 of arranging towers from the smallest to the greatest is omitted, then the number of towers that can be constructed increases and, consequently, the number of additive decompositions of the number 10 in two integers increases by four. Indeed, the first four towers in Figure 4.1 can be turned into two towers and, therefore, the number of ways the number 10 can be presented as the sum of two integers in all possible ways (taking into account the commutative property of addition) is nine.

Remark 2. A more difficult question is to find the number of ways the number 10 can be decomposed into a sum of three integers in all different orders. Figure 4.2 can be used to explore how many new towers stem from each of the eight towers. To this end, one has to recognize an important difference in the towers. Namely, either a set of three towers has them all of different height or two of the towers are of the same height. In the latter case, the tallest tower can be positioned among two identical towers in three different ways. In the former case, counting is more complicated. As example, consider the second triple of towers. If the tallest tower is located at the far right, then the other two towers can be positioned in two ways. If the tallest tower is positioned at the far left, then the remining two towers can also be positioned in two ways. Finally, the tallest tower can be positioned in the middle. In that case, the other two towers can also be positioned in two ways. This reasoning is explained in Figure 4.5. Therefore, according to Figure 4.2, we have four triples of the former type and four triples of the latter type. From each of the four triples of the former type three towers result, giving the total of 12 towers. From each of the four triples of the latter type six towers result, giving the total of 24 towers. In all, we have 36 towers. In other words, the number 10 can be decomposed into the sums of three integers in all possible orders in 36 ways.

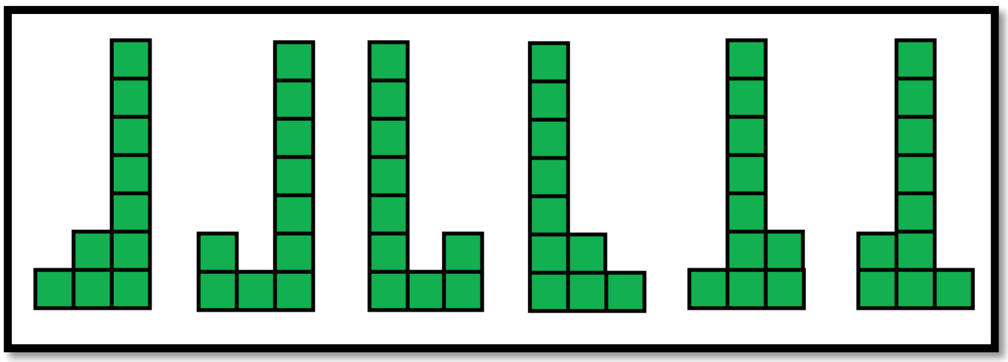

Figure 4.5. Counting the number of permutations of three different towers.

CHAPTER 5: ACTIVITIES WITH ADDITION AND MULTIPLICATION TABLES

> "Well-prepared beginning teachers of mathematics have solid and flexible knowledge of core mathematical concepts and procedures they will teach, along with knowledge both beyond what they will teach and foundational to those core concepts and procedures" (Association of Mathematics Teacher Educators, 2017, p. 8).

5.1 Geometrization of addition and multiplication tables

The multiplication table (already discussed in Chapter 2) is well known as a part of elementary school mathematics curriculum. Its traditional educational uses are purely arithmetical stemming from recording multiplication facts which, in turn, emerge from repeated addition. Some more conceptually-oriented activities with the multiplication table were presented in Chapter 2. Visually, repeated addition can be represented through the concrete activity of the construction of rectangles by using unit squares. The product of two integers can be understood as the number of the squares inside a rectangle so constructed and, therefore, conceptually, this number can be interpreted as the area of the rectangle expressed in terms of the number of unit squares used to measure area of the rectangle. Another geometric characteristic associated with a rectangle is perimeter. In the context of a rectangle filled with unit squares, its perimeter can be defined as the total number of outer edges of the squares forming the border of the rectangle. Just as in the case of area, one can consider only a pair of adjacent sides as the essential characteristics of a rectangle and the number of edges forming such pair of sides is one-half of the perimeter often called semi-perimeter. The sums of two integers can form the addition table. Therefore, using the concepts of area and semi-perimeter, both the multiplication and the addition tables can be explored within a unified context of arithmetic and geometry.

5.2 Counting numbers with special properties in addition/multiplication tables.

Numeric tables represent a conceptually rich milieu of activities for fostering mathematical reasoning skills using a numerical approach and serving as a springboard into algebra. One activity of that kind deals with answering the following questions: *What is the sum of all numbers in the 10x10 addition/multiplication table? How many even numbers are there in the 10x10 addition/multiplication table? What is the sum of all even numbers in the 10x10 addition/multiplication table?*

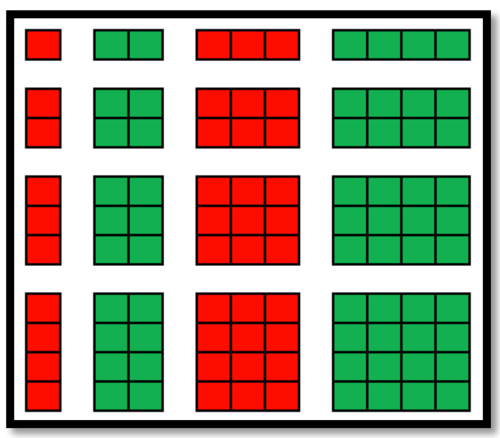

Figure 5.1. There are 16 types of rectangles ranging in size from 1×1 to 4×4.

Before answering the above and like questions, let us create two tables with all types of products and sums ranging in size from 1×1 to 4×4 and find the areas and semi-perimeters for all the rectangles. In that case, we have 1×1, 1×2, 1×3, 1×4, 2×1, 2×2, 2×3, 2×4, 3×1, 3×2, 3×3, 3×4, 4×1, 4×2, 4×3, 4×4 products and 1 + 1, 1 + 2, 1 + 3, 1 + 4, 2 + 1, 2 + 2, 2 + 3, 2 + 4, 3 + 1, 3 + 2, 3 + 3, 3 + 4, 4 + 1, 4 + 2, 4 + 3, 4 + 4 sums forming the corresponding multiplication and addition tables. Figure 5.1 shows all 16 types of the rectangles related to the above products and sums. The corresponding multiplication and addition tables are shown in Figures 5.2 and 5.3, respectively.

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

Figure 5.2. A 4×4 -multiplication table created by *Wolfram Alpha*.

+	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

Figure 5.3. A 4×4 -addition table created by *Wolfram Alpha*.

How can one find the sum of numbers in the 4×4 multiplication table shown in Figure 5.2? One way to find the sum is pretty straightforward: to find the sum of numbers in each of the four rows (or columns). We have 1 + 2 + 3 + 4 = 10, 2 + 4 + 6 + 8 = 20, 3 + 6 + 9 + 12 = 30, 4 + 8 + 12 + 16 = 40, so that the sum is equal to 10 + 20 + 30 + 40 = 100.

The first way of finding the sum can prompt the idea that the sum in each row (column) is a multiple of the sum in the first row (column) where the multiplier is equal to the row's (column's) number. That is,

$$(1 + 2 + 3 + 4\,) + (2 + 4 + 6 + 8) + (3 + 6 + 9 + 12) + (4 + 8 + 12 + 16)$$

$$= 1\times(1+2+3+4\,) + 2\times(1+2+3+4\,) + 3\times(1+2+3+4\,) + 4\times(1+2+3+4\,)$$

$$= 1\times(1+2+3+4\,)\ \times(\ 1+2+3+\ \ ) = 10\times 10 = 100.$$

The third way is less obvious than the previous two ways. It requires insight. To this end, one can do summation within gnomons as follows:

$$1 + (2 + 4 + 2) + (3 + 6 + 9 + 6 + 3) + (4 + 8 + 12 + 16 + 12 + 8 + 4) = 1 + 8 + 27 + 64 = 100.$$

Through this summation one can note that $1 = 1^3$, $8 = 2^3$, $27 = 3^3$, $64 = 4^3$, and, therefore,

$$1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2.$$

One can see that insight is not just another way of solving a problem. It is insight that allowed for the equality relationship between the sum of cubes of the first four counting numbers and the square of their sum to be discovered.

How many even numbers are there in the 4×4 multiplication table? In an even row (column) of the table all numbers are even; in an odd row of the table, only half of numbers are even. Therefore, out of 16 numbers of the table, 12 numbers are even, and 4 numbers are odd. In order to find the sum of even numbers in the 4×4 multiplication, one can first find the sum of odd numbers in the table. This sum is equal to (1 + 3) + 3(1 + 3) = (1 + 3)(1 + 3) = 16. Therefore, the sum of even numbers in the table is equal to 100 – 16 = 84.

By analogy, one can find the sum of numbers in the 10×10 multiplication table. To this end, the expression $(1 + 2 + 3 + 4)^2$ used to find the sum of numbers in the 4×4 table can be extended to the form $(1 + 2 + 3 + ...+10)^2$. Now, we can apply the results of two conceptual shortcuts to find the value of the last expression.

First, $1 + 2 + 3 + ...+10 = 1/2\ [(1 + 2 + 3 + ...+10) + (10 + 9 + 8 + ... + 1)] = 10 \cdot \frac{10}{2} = 55$.

Second, 55^2 can be represented through the square shown in Figure 5.4 from where it follows that $55^2 = 50^2 + 2 \cdot 50 \cdot 5 + 5^2 = 2500 + 500 + 25 = 3025$.

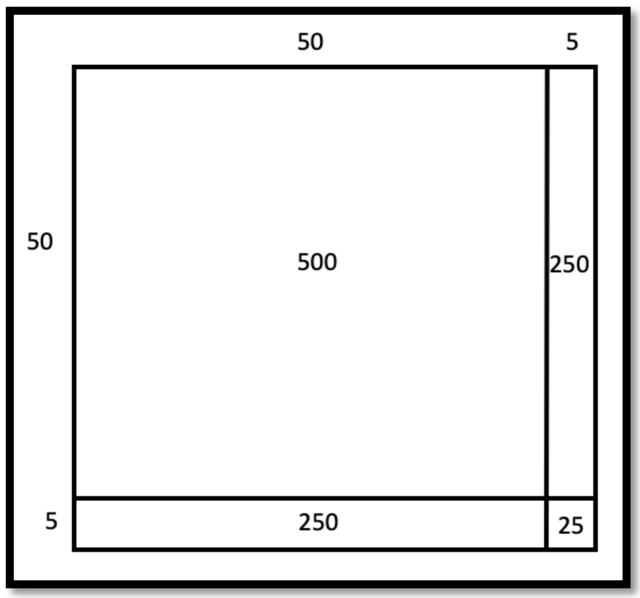

Figure 5.4. Finding 55^2 through a conceptual shortcut.

CHAPTER 6: USING TECHNOLOGY IN POSING PROBLEMS

> "Well-prepared beginning teachers of mathematics ... regard *doing mathematics* as a sense making activity that promotes ... problem posing" (Association of Mathematics Teacher Educators, 2017, p. 9).

The use of technology in mathematical problem posing can be characterized as a cultural support of designing new curriculum materials for a mathematics classroom. Through this creative process, one can use tools of technology developed for various practical and scholastic purposes. For example, a spreadsheet, originally developed for non-educational purposes, was retrofitted to be used in education, including mathematics education. In particular, in the specific context of mathematical problem posing, one can put to work the power of a computational tool in order to generate a solution (which may not be unique) to a grade-appropriate problem. A spreadsheet can be programmed in such a way that a numeric problem can be both posed and solved. This implies that computing technology, in general, makes problem posing and problem solving being inherently linked to each other. The important task for a teacher candidate is to appreciate this relation between problem posing and problem solving, to learn how to recognize the emergence of a new problem and to "be expert in ... the craft of task design" (Conference Board of the Mathematical Sciences, 2012, p. 65). Such recognition should be developed both in a computational and non-computational learning environment.

In order for technology to have a positive effect on problem posing, one should not only know how to use it but, more importantly, how to interpret the results that a technology tool generates. This interpretation requires understanding of what may be called the didactical coherence of a problem. The paper (Abramovich and Cho, 2008) is available on-line (https://sie.scholasticahq.com/issue/791) where the meaning of this term is explained in detail. In order to illustrate the use of a spreadsheet in problem posing, consider

Problem. It takes 51 cents in postage to mail a letter. A post office has stamps of denomination 5 cents, 7 cents, and 10 cents. How many combinations of the stamps could Anna buy to send a letter if the order in which the stamps are arranged on an envelope does not matter?

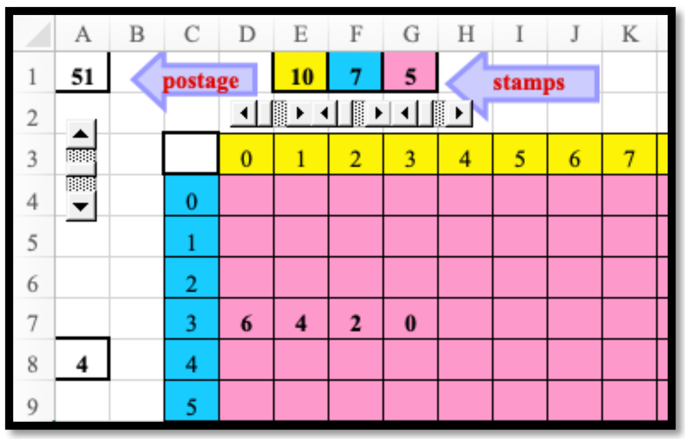

Figure 6.1. Making 51-cent postage out of three stamps.

Solution. The spreadsheet shown in Figure 6.1 is designed as follows. In cells E1, F1, and G1 the denominations of available stamps are displayed. In cell A1 the required postage is displayed. The ranges [D3:K3] and [C4:C9] include possible quantities of the 10-cent and 7-cent stamps, respectively. In the (two-dimensional) range [D4:K9] the quantities of the corresponding 5-cent stamps are generated by the spreadsheet through a specially designed formula (not included here) which solves the equation $10x + 7y + 5z = 51$, where x, y and z are possible quantities of the available stamps that make the required postage. One can see (range [D7:G7]) four values of the quantities of the smaller denomination stamp. For example, the triple (1, 3, 4) with the elements in cells E3, C7, E7, respectively, indicates that with one 10-cent stamp, three 7-cent stamps, and four 5-cent stamps, the required postage, 51 cents, can be achieved. One can see that all four solutions include three 7-cent stamps. This observation can be used to explain how the problem can be solved by a student using paper and pencil. Indeed, using only 10-cent and 5-cent stamps, the 51-cent postage cannot be made (as 51 is not a multiple of 5) and, therefore, one has to try using a 7-cent stamp also. Furthermore, only three 7-cent stamps add up to a value which, when subtracted from 51 yields a multiple of 5, namely 30. The possible representations of 30 through 10 and 5 are:

$30 = 10 + 10 + 10$; $30 = 10 + 10 + 5 + 5$; $30 = 10 + 5 + 5 + 5 + 5$, and $30 = 5 + 5 + 5 + 5 + 5 + 5$. These representations correspond to the triples (3, 3, 0), (2, 3, 2), (1, 3, 4), and (0, 3, 6). This is an example of how a teacher can use technology not only to design a problem for their students but also learn how to solve the problem in a systematic way without using technology. By changing the entries in the cells A1, E1, F1, and G1 and their contextual meaning, new problems can be posed and solved within the spreadsheet. For example (see Chapter 11, section 11.2)., one can find the number of ways a quarter (cell A1) can be changed into demes (cell E1), nickels (cell F1), and pennies (cell G1). Note that the spreadsheet is designed in such a way that the smallest numeric data should always be entered in cell G1.

CHAPTER 7: FRACTIONS

> "In grade 3, instructional time should focus on ... developing understanding of fractions, especially unit fractions" (Common Core State Standards, 2010, p. 21).

7.1 Two contexts for fractions

There are two major real-life contexts leading to the concept of fraction as an extension of integer arithmetic when the division of two integers does not results in an integer: the part-whole context and the dividend-divisor context. Recall that the operation of division was introduced through two modeling contexts: measurement context and partition context (Chapter 2, section 2.4). One can say that the part-whole context extends the measurement context and the dividend-divisor context extends the partition context. Consider the relation $6 \div 2 = 3$ which can be interpreted as measuring six donuts by two donuts as a way of creating three two-donut servings. Alternatively, this division of six by two can be interpreted as partitioning six donuts between two people evenly, allowing each person to receive three donuts. But what about dividing ten donuts among four people? While this is not impossible to do, fractions, typically are introduced in the context of dividing a smaller number by a larger number. With this in mind, consider the case of dividing 3 by 4. As an extension of integer arithmetic, one can interpret this operation as measuring 3 donuts by 4 donuts resulting in the number 3/4 which, in the form of abstraction, shows how many times the number 3 includes the number 4. Alternative interpretation of dividing 3 by 4 is partitioning 3 (identical) donuts among 4 people evenly which can physically demonstrate that each person would get 3/4 of a donut. Indeed, each of the three donuts can be divided in four parts and the resulting twelve pieces are then partitioned among four people so that each person gets three pieces of one-fourth of a donut. The former context of division is called the part-whole context and the latter context of division is called the dividend-divisor context.

Typically, the dividend-divisor context, although appearing as less abstract, is omitted from teaching fractions in the schools. A reason for such omission could be due to the fact that the part-whole context is already introduced through dividing an object (e.g., a rectangular-shaped pie) into several equal parts; for example, one pie divided in two equal parts, each of which is called one-half. At the same time, it is important for elementary teachers to appreciate connection between fractions and division (Conference Board of Mathematical Sciences, 2012).

Often, when talking about three (identical) pies being equally divided among five people (with three being the dividend and five being the divisor), instead of recognizing an opportunity for the dividend-divisor context to be introduced, a teacher might say in passing that each person would get 3/5 of a pie without paying attention to the significance of this context as perhaps the most natural way of extending integer arithmetic to that of fractions when divided is smaller than divisor. Such cases explain the following assumption about mathematics teacher preparation: "Teaching mathematics effectively requires career-long learning ... [and] intentional efforts to seek additional knowledge ... [to be used] in supporting students' learning of mathematics" (Association of Mathematics Teacher Educators, 2017, p.1).

The dividend-divisor context makes it possible to explain why a fraction has multiple representations through equivalent fractions and an integer has only one representation through itself. For example, 3/5 = 6/10 = 9/15 = 12/20 = …, and 3 has only one self-representation, 3 = 3. But if one recognizes that in the fraction notation the operation of division is hidden (that is, it is only identified, by presenting the dividend and the divisor, yet not completed), then at least two things become clear: there are many ways to represent three through an operation (e.g., $3=2+1=5-2=1\cdot 3=12\div 4$), and dividing three (identical) things in five equal parts gives the same result (i.e., the same quantity of material out of which the things are comprised) when six such things are divided into ten equal parts, nine – into fifteen, and so on.

One can introduce the fraction 3/5 by extending partition model for division to non-integer arithmetic; for example, by dividing three identical objects such as pies among five people. As shown in Figure 7.1, each of the three identical rectangles (representing pies) is divided into five equal parts. Through this process, fifteen equal size pieces result. Dividing 15 pieces among five people through the partition model for division yields three pieces of pie for each person, a piece being 1/5 of the whole pie. Thus, we have the fraction 3/5 representing the operation $3 \div 5$, the outcome of which has only been identified through the notation 3/5, yet not completed. Connection between a fraction and division is mentioned by the Conference Board of the Mathematical Sciences (2012) as one of the most fundamental ideas in arithmetic that serves as a numeric characteristic of the ratio concept in middle school mathematics when the dividend-divisor comparison of the elements of two sets yields the same fraction. As was already mentioned, dividing three (identical) pies among five people results in the same quantity of a pie as dividing six such pies among ten people. That is, in both cases the ratio (see Chapter 9) of pies to people is the same, expressed through the fraction 3/5. At the same time, the fraction 3/5 can be understood within the context of dividing 15 pieces of pie among five people (through partition model for division) as repeated addition of the unit fraction 1/5 in the form $\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$.

The teaching of fractions conceptually can be enhanced by the use of the so-called tape diagrams (Common Core State Standards, 2010) aimed at explaining formal operations and their meaning. Just as the teaching of writing was recommended to "be arranged by shifting the child's activity from drawing things to drawing speech" (Vygotsky, 1978, p. 115), the teaching of arithmetic of fractions (including whole numbers) can be arranged as a transition from drawing a physical meaning of addition, subtraction, multiplication and division to describing the visual and the physical through culturally accepted mathematical notation. With this in mind, the W^4S principle can be used in the teaching of fractions. The diagram of Figure 7.1 is a simple example of explaining the meaning of the fraction 3/5 by drawing an image of dividing three objects in five equal parts each.

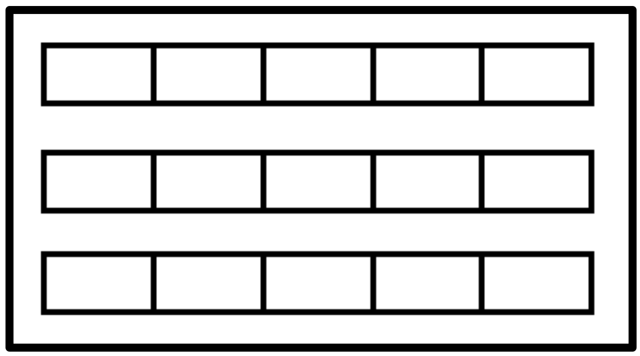

Figure 7.1. Dividing 5 into 3.

7.2 Adding and subtracting fractions

In order to develop skills in adding fractions, one can begin with reflecting on the addition of integers. How do we add integers? For example, the meaning of the operation 2 + 3 stems from the context of adding two of something to three of something and the result is 5 (of something). But what if we add two pears and three bananas? The resulting number 5 represents neither pears nor bananas because the addends belong to different denominations. Is there a denomination to which both pears and bananas belong? The word fruit (or just thing) may be an answer to this question: adding two pears and three bananas yields five fruits (things). In the operation 2 + 3 both addends are abstractions, de-contextualized from denominations they represent. But in this de-contextualization, there is one concrete common characteristic – both 2 and 3 are comprised of the *same* unit of measurement. So, three units being added to two units yield five units. This unit is a fruit decontextualized from reality and abstracted to become a unit of measurement.

Similar situation is with fractions. Both 1/2 and 2/3 are fractions (parts) of the same unit. However, the situation with fractions is more difficult than with integers because this something is both a concrete thing and a unit of measurement through which both fractions can be measured. So, talking about 1/2 and 2/3 of a pie (which is the unit), we have the two fractions of the pie to measure by another, fractional unit. In doing so, 1/2 and 2/3 of the pie are measured by 1/6 of the pie to have the total 7/6 of the pie. That is, de-contextualization yields

$$\frac{1}{2}=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{3}{6},\ \frac{2}{3}=\frac{1}{3}+\frac{1}{3}=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{4}{6} \text{ and } \frac{1}{2}+\frac{2}{3}=\frac{7}{6}.$$

The same reasoning can be applied to the operation 2/3 – 1/2. Indeed, 2/3 – 1/2 = 4/6 – 3/6 = 1/6, where the notations 4/6 and 3/6 are understood as 1/6 repeated four and three times, respectively.

Four different types of fractions can be identified: a unit fraction (e.g., $\frac{1}{3}$ – a fraction with numerator one), a proper fraction (e.g., $\frac{2}{7}$ – a fraction with numerator smaller than denominator), an improper fraction (e.g., $\frac{7}{2}$ – a fraction with denominator smaller than numerator), and a mixed fraction (e.g., $5\frac{2}{3}$ but not $5\frac{3}{2}$). Indeed, $5\frac{2}{3}$ is a notation used to represent the sum of the integer part of a (rational) number and its fractional part, $5\frac{2}{3}=5+\frac{2}{3}$ (thus $5\frac{3}{2}$ is not a notation used to represent a mixed fraction as $\frac{3}{2}$ itself has a non-zero integer part). In order to turn a mixed fraction into an improper fraction, that is, to add an integer and a fraction, one has to find a common measure for the two numbers. To this end, one has to turn an integer into a fraction and add two fractions. In our case, $5\frac{2}{3}=5+\frac{2}{3}=\frac{5\cdot 3}{3}+\frac{2}{3}=\frac{5\cdot 3+2}{3}=\frac{17}{3}$. This explains conceptually a commonly known procedure of turning a mixed fraction into an improper fraction: $5\frac{2}{3}=\frac{5\cdot 3+2}{3}=\frac{17}{3}$.

The next two operations to be discussed in the context of fractions are multiplication and division. A common fraction can be understood both from the perspectives of multiplication and division. Indeed, the fraction $\frac{a}{b}$ can be understood both as the unit fraction $\frac{1}{b}$ repeated a times and as the integer a divided by the integer b. Multiplication and division of fractions can be introduced by using area model for fractions, one of the three models for teaching and learning fractions: area, measurement, and set models. Conceptual meaning of the reduction of a fraction to the simplest form (something that was not observed in the context of whole number arithmetic) and the "invert and multiply" rule commonly utilized for the division of fractions will be discussed as well. A number of word problems that provide different real-life situations for modeling mathematics with fractions will be presented.

7.3 Multiplying two proper fractions

Where does the rule of multiplying two fractions come from? What is this rule? How, for example, can one justify, conceptually, that $\frac{2}{5}\cdot\frac{3}{7}=\frac{2\cdot 3}{5\cdot 7}=\frac{6}{35}$? More specifically, why does the operation of multiplication compel multiplying numerators and denominators of two fractional factors? The last question is consistent with expectations of the Common Core State Standards (2010) for students in Grade 5 to "use the meaning of fractions, of multiplication and division ... to understand and explain why the procedures for ... fractions make sense" (p. 33). One can begin explanation from the multiplication of integers where $2 \cdot 3$ is understood as repeating three things twice, or, alternatively, taking two (identical) groups of any three objects. In the case of multiplying two fractions, repeating a fraction fractional number of times does not make sense unless this abstraction is put in context (contextualization) and explained through a two-dimensional diagram as shown in Figure 7.2. Here, the large rectangular grid represents one whole, 3/7 of which are marked with x's and 2/5 of which are marked with 0's.

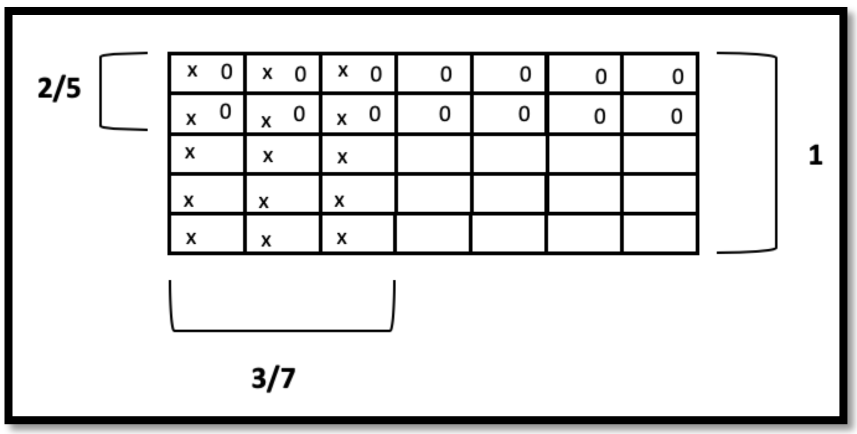

Figure 7.2. Finding the product $\frac{2}{5} \cdot \frac{3}{7}$ within a grid.

Which part of the whole does the product represent? Could the product be represented by a region being larger than the whole? The answers to these questions are in the meaning of multiplication that does not change as the number system changes; that is, a physical meaning of multiplication is the same for objects described by integers as for those described by fractions. How can one contextualize the product $2 \cdot 3$? This product represents the number of objects included in two groups *of* three objects. In other words, the multiplication sign is characterized contextually by the preposition 'of'; that is, multiplying two quantities means taking a certain quantity *of* another quantity. Likewise, $\frac{2}{5}\cdot\frac{3}{7}$ means taking $\frac{2}{5}$ *of* $\frac{3}{7}$; that is, taking the quantity 2/5 *of* the quantity 3/7. In order to make this operation more concrete; in other words, in order to demonstrate the skill of contextualization of the operation, one can first take $\frac{3}{7}$ of the whole (e.g., of a pie) and then take $\frac{2}{5}$ of the piece taken. This action results in a new fraction of the whole characterized by the product $\frac{2}{5}\cdot\frac{3}{7}$. This is a region where both marks, 0's and x's, overlap after 3/7 are marked with x's and 2/5 are marked with 0's. Now one can see the region described by the product. Yet, the question remains: how can one describe the region with both marks through a single fraction? The overlap of x's and 0's can be seen as 2 groups of 3 objects; that is, $2\cdot 3=6$ cells belong to the overlap. The total number of cells in the whole can be seen as 5 groups of 7 objects; that is, $5\cdot 7=35$ cells comprise the whole grid. Therefore, by moving from visual to symbolic, that is, through de-contextualization, the overlap as a fraction of the grid (the whole) can be expressed numerically as $\frac{6}{35}$. In other words, $\frac{2}{5}\cdot\frac{3}{7}=\frac{2\cdot 3}{5\cdot 7}=\frac{6}{35}$. Alternatively, $\frac{2}{5}\cdot\frac{3}{7}=(\frac{1}{5}+\frac{1}{5})\cdot(\frac{1}{7}+\frac{1}{7}+\frac{1}{7})=\underbrace{\frac{1}{5}\cdot\frac{1}{7}+...+\frac{1}{5}\cdot\frac{1}{7}}_{six\ times}=6\cdot(\frac{1}{5}\cdot\frac{1}{7})$. But contextually, the product $\frac{1}{5}\cdot\frac{1}{7}$ is understood as taking 1/5 of 1/7, something that, as shown in Figure 7.3, is equal to 1/35. Thus, $6\cdot(\frac{1}{5}\cdot\frac{1}{7})=6\cdot\frac{1}{35}=\underbrace{\frac{1}{35}+...+\frac{1}{35}}_{six\ times}=\frac{6}{35}$.

In general, $\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$. That is, the rule (algorithm) of multiplying fractions has been developed conceptually by assisting one in seeing "where a mathematical rule comes from"

(Common Core State Standards, 2010, p. 4). This position is consistent with a more recent statement by the Association of Mathematics Teacher Educators (2017) that “mathematics teaching ... requires not just general pedagogical skills but also content specific knowledge, skills, and dispositions” (p. 2). These mathematics education positions are quite different from “early schooling practices ... [when the] “rule method” (memorize a rule, then practice using it) was the sole approach used in U.S. arithmetic textbooks from colonial times until the 1820s” (Conference Board of the Mathematical Sciences, 2012, p. 9).

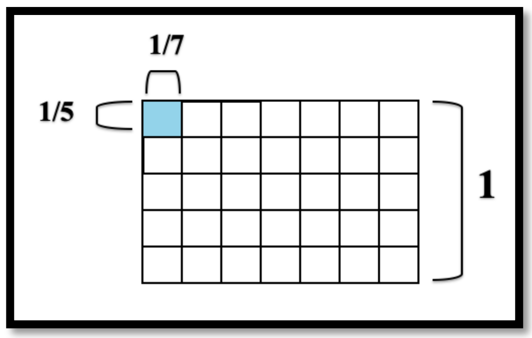

Figure 7.3. Taking 1/5 of 1/7.

7.4 Multiplying two improper fractions

In order to find out when the product of two fractions is represented by a region larger than the whole, consider the case of multiplying two improper fractions using a grid. How can one construct an image of the product $\frac{5}{2} \cdot \frac{7}{3}$ proceeding from an image representing one whole? As in the case of multiplying proper fractions, we start with drawing a rectangle (the borders of which are solid lines) to represent the whole (Figure 7.4). The next step is to divide (vertically) the rectangle (with the solid borders) into three equal parts, each of which is 1/3 of the rectangle and then extend it to the right by another four thirds to get 7/3 (marked with 0’s). Then, the so constructed fraction 7/3 is divided (horizontally) into two equal parts each of which is 1/2 of 7/5, and then extend it down by another three halves to get 5/2 of 7/3, that is, the product $\frac{5}{2}\cdot\frac{7}{3}$.

One can see that the cells marked with x's represent 5/2 of the whole (the rectangle with the solid borders) and the cells marked with 0's represent 7/3 of the whole. The cells marked with only x's and with only 0's represent, respectively, 3/2 and 4/3 (of the whole). The cells with no marks represent the product $4\cdot\frac{1}{2}$, which (by using 2 cells as a new unit) can be reduced to the integer 2 as the simplest form of the fraction 4/2. Put another way, the diagram of Figure 7.4 reflects the distributive property of multiplication over addition:

$$\frac{5}{2}\cdot\frac{7}{3}=(2+\frac{1}{2})(2+\frac{1}{3})=2\cdot 2+2\cdot\frac{1}{3}+\frac{1}{2}\cdot 2+\frac{1}{2}\cdot\frac{1}{3}=4+1+\frac{2}{3}+\frac{1}{6}=5+\frac{4}{6}+\frac{1}{6}=5+\frac{5}{6}=5\frac{5}{6}.$$

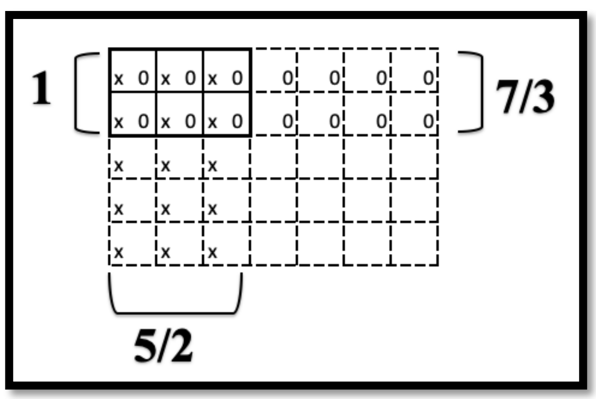

Figure 7.4. The product $\frac{5}{2}\cdot\frac{7}{3}$ as the sum $1+\frac{4}{3}+\frac{3}{2}+2$ of four non-overlapping regions.

Finally, one can see that the overlap of the fractions 5/2 and 7/3, that is, the part of the diagram of Figure 7.4 with both marks, coincides with the whole. Unlike the case of the product of two proper fractions, when their overlap is their product located within the whole, the overlap of the product of two improper fractions, as shown in Figure 7.4, is the whole itself. Furthermore, the equality $\frac{5}{2}\cdot\frac{7}{3}=\frac{35}{6}$ means that the product consists of 35 cells each of which has the name given by the whole which is comprised of six such cells.

Likewise, the product $\frac{5}{2}\cdot\frac{3}{7}$ in which only one factor is represented by an improper fraction is represented by a region that is larger than one whole (in Figure 7.5 the border of the whole is made of solid lines). Of course, not every product of a proper and an improper fraction is greater than one. The product $\frac{5}{2}\cdot\frac{3}{7}$ is greater than one. Whereas the product of two proper or two improper fractions is, respectively, smaller or greater than the whole, the product of two fractions of different kind may be greater or smaller than the whole. One can see (Figure 7.5) that the product $\frac{5}{2}\cdot\frac{3}{7}=\frac{3}{7}\cdot\frac{5}{2}$ (it is easier to see how one can take 3/7 of 5/2) is the sum of two non-overlapping regions: the regions with both x's and 0's and the region below it with only 0's.

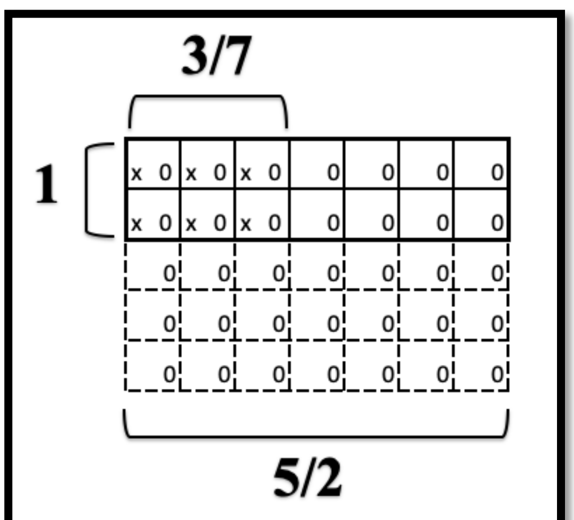

Figure 7.5. The product $\frac{3}{7}\cdot\frac{5}{2}$ as the sum of two non-overlapping regions: $\frac{6}{14}$ and $\frac{9}{14}$.

7.5 Dividing fractions

Multiplication and division (just as addition and subtraction) are operations in the relation of reciprocity. As was already observed with integers, division is a more complicated operation than multiplication. The major difference between the two operations is that whereas the product of two integers is an integer, their quotient may bring with it a non-zero remainder. However, the set of rational numbers (that includes negative numbers not discussed here) is closed for all four arithmetical operations, except for the case of a zero divisor. So, both the product and the quotient of two fractions are fractions (except the zero divisor). The goal of this section is to demonstrate how division of fractions can be carried out conceptually using (two-dimensional) area model for fractions just as it was carried out when multiplying fractions.

7.5.1 Dividing whole numbers

To begin, consider the case of dividing whole numbers by using area model. What does it mean to divide 5 into 4 (alternatively, dividing 4 by 5)? Because $5 > 4$, the smaller number cannot be measured by the larger number as one of the interpretations of division suggests (unless we develop an extended interpretation of the division of integers). That is, in terms of the measurement model for division, the inclusion of 5 into 4 is an abstraction, as something larger may not be physically included in its part. Nonetheless, this abstraction can be represented in the form of a (positive) fraction smaller than one. This fraction also stems from the partition model for division in the context of dividing 4 by 5, something that, through an appropriate contextualization seems to be more concrete. The partition model for division in this context means dividing four things among five people in a fair way. Obviously, one should be able to cut things into equal pieces; that is, the things to be cut should be conducive to be partitioned in smaller (equal) parts. For example, it is not possible to divide four marbles in five equal parts. Yet, it is possible to make five servings from four (identical) pies. To do that, one has to divide each pie into five equal parts, so that a serving is 4/5 of a pie. Put another way, in order to find how many times 5 is included into 4 one has to divide 4 by 5. (In particular, this shows connection between fractions and division). How can one carry out this division physically (e.g., using a picture)?

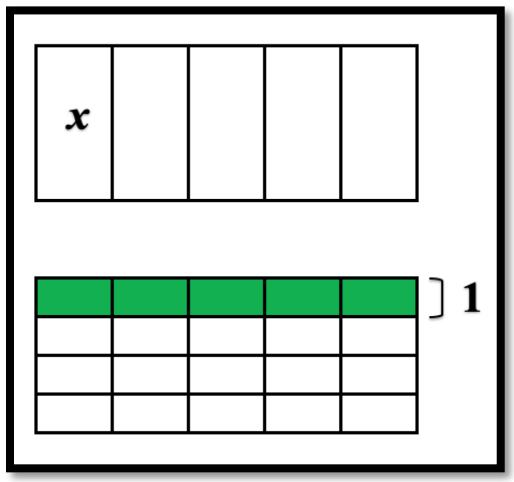

Figure 7.6. Using two-dimensional model when dividing 5 into 4.

Let $4 \div 5 = x$. Then x is a missing factor in the equation $5x = 4$. The top part of Figure 7.6 shows the left-hand side of the last relation, that is, $5x$. At the same time, the bottom part of Figure 7.6 represents the right-hand side of the equation, that is, 4. Consequently, the unity is shaded dark and it is one-fourth of the top part of Figure 7.6. The two-dimensional area model shows that because the unity consists of five cells, each of which represents the fraction $\frac{1}{5}$, previously unknown x becomes known as it consists of four such cells. That is, $x = \frac{4}{5}$ or $4 \div 5 = \frac{4}{5}$.

7.5.2 Dividing proper fractions

Now, using the same method, let us divide two proper fractions. For example, let us find the value of $\frac{4}{5} \div \frac{3}{4}$ using area model for fractions. The result of this division is a number *x*, three-fourth of which is equal to four-fifth; that is, *x* is the missing factor in the equation $\frac{3}{4}x = \frac{4}{5}$. Figure 7.7 includes three diagrams, the far-left one representing *x*. The diagram in the middle of this figure shows that *x* consists of seven equal sections with an unknown numerical value. In order to make it known, one has to represent the unity. Because $\frac{3}{4}x$ (shown in the middle of Figure 7.7) is equal to $\frac{4}{5}$ (of the unity), it has to be extended by $\frac{1}{5}$ in order to show the unity. This extension is shown in the far-right part of Figure 7.7 using dotted lines. Through this method, the unity turned out being divided into $3 \cdot 5 = 15$ equal sections each of which is 1/15 of the unity. At the same time, we see that *x* comprises $4 \cdot 4 = 16$ such sections. That is, $x = \frac{16}{15}$. Put another way, in order to divide $\frac{4}{5}$ by $\frac{3}{4}$ one has to multiply $\frac{4}{5}$ by the reciprocal of $\frac{3}{4}$; that is, $\frac{4}{5} \div \frac{3}{4} = \frac{4}{5} \cdot \frac{4}{3} = \frac{16}{15}$. Below, the meaning of this rule, commonly referred to as *Invert and Multiply*, will be explained.

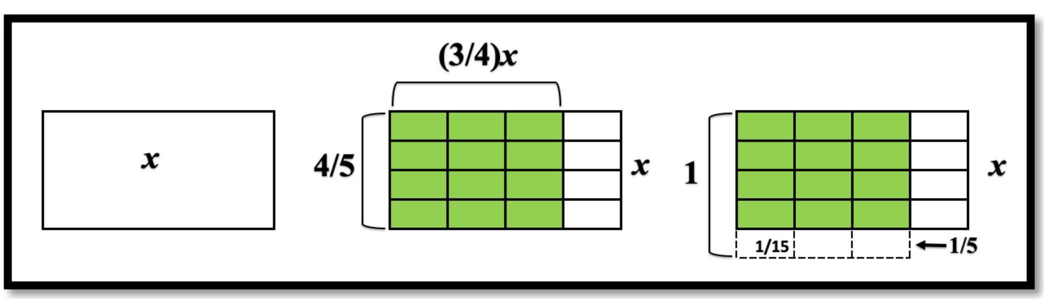

Figure 7.7. Using two-dimensional method when dividing $\frac{4}{5}$ by $\frac{3}{4}$.

7.6 Conceptual meaning of the *Invert and Multiply* rule

The *Invert and Multiply* rule is used when dividing two fractions and replacing division by multiplication as an operation which turns the product of two integers into an integer. For example, in the case $\frac{4}{7} \div \frac{3}{5}$ it might be tempting, by analogy with multiplying two fractions when their numerators and denominators are multiplied, to proceed dividing numerators and denominators to have $\frac{4 \div 3}{7 \div 5}$. This, however, brings us back to the division of the original two fractions by using the dividend-divisor context for division in the case of fractions. The *Invert and Multiply* rule allows one not to deal with the fraction $\frac{4 \div 3}{7 \div 5}$, but rather to compute the fraction $\frac{4 \cdot 5}{7 \cdot 3}$ in which the numerators are multiplied by demominators.

7.6.1 The case of dividing integers

In order to understand conceptual meaning of the *Invert and Multiply* rule, one has to show how it works in the context of the division of two whole numbers. For example, the division fact $8 \div 4 = 2$ can be interpreted as finding how many 4-tile towers can be built out of 8 tiles (Figure 7.8). Here, the numbers 4 and 8 point at a tile as the unit, and, therefore, measuring 8 tiles by 4 tiles in the process of building towers yields 2 towers. If division is replaced by multiplication and 4 is replaced by 1/4 (the inversion of 4), we have the relation $8 \cdot \frac{1}{4} = 2$. This time, we have eight one-fourths; that is, tile is not the unit anymore, but it is 1/4 of a tower, which became a new unit through using the *Invert and Multiply* rule. Repeating one-fourth of a tower eight times yields two towers. That is, the *Invert and Multiply* rule makes sense in the case of division of two whole numbers when the measurement model is used.

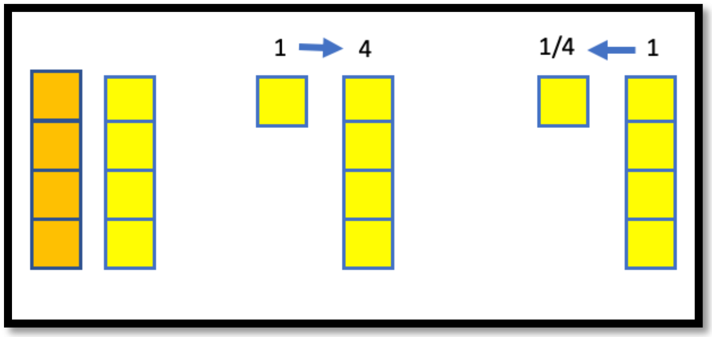

Figure 7.8. Building four-tile towers out of eight tiles: $8 \div 4 = 8 \cdot \frac{1}{4}$.

However, if one has to build four identical towers out of eight tiles, we also have this process described as $8 \div 4 = 2$, but this time, we use partition model for division. Replacing $8 \div 4$ by $8 \cdot \frac{1}{4}$ does not result in the change of unit as 8 and 4 are comprised of different units – tiles and towers. Due to commutative property of addition, $8 \cdot \frac{1}{4} = \frac{1}{4} \cdot 8$ and therefore, $8 \div 4$ is replaced by taking one-fourth of eight tiles. This yields two tiles for a tower. So, in the case of partition model for division the change of unit does not take place through the *Invert and Multiply* rule. Instead, the meaning of the *Invert and Multiply* rule in the case of partition model for division can be understood in terms of the dividend-divisor context.

7.6.2 The case of dividing fractions

Now, in order to explain the *Invert and Multiply* rule in the context of fractions, consider the division operation $3\frac{1}{3} \div \frac{5}{6}$. The operation can be contextualized and interpreted as measuring the distance of $3\frac{1}{3}$ miles by the segment of length $\frac{5}{6}$ miles when a truck covers any segment of this length in a minute (assuming the uniform movement). This measuring would result in the number of minutes the driver of the truck needs to cover this distance. Here, one mile is the unit, and the truck makes $\frac{5}{6}$ of the unit per minute. Replacing division by multiplication and $\frac{5}{6}$ by $\frac{6}{5}$ yield $3\frac{1}{3}\cdot\frac{6}{5}=\frac{10\cdot 6}{3\cdot 5}=4$. This time, the unit is the segment ran by the truck in a minute; this segment is smaller than 1 mile. Through this process, 1 mile becomes 6/5 of the segment and the later becomes a new unit (Figure 7.9). If mile is a unit, then the segment, run in a minute, is 5/6 (miles). If a segment is a unit, then one mile is 6/5 (of the segment). Having $3\frac{1}{3}$ miles means that $\frac{6}{5}$ of this segment (that is, 1 mile) when repeated $3\frac{1}{3}(=\frac{10}{3})$ times yields 4 minutes. Note that in the described case, the measurement model for division was used.

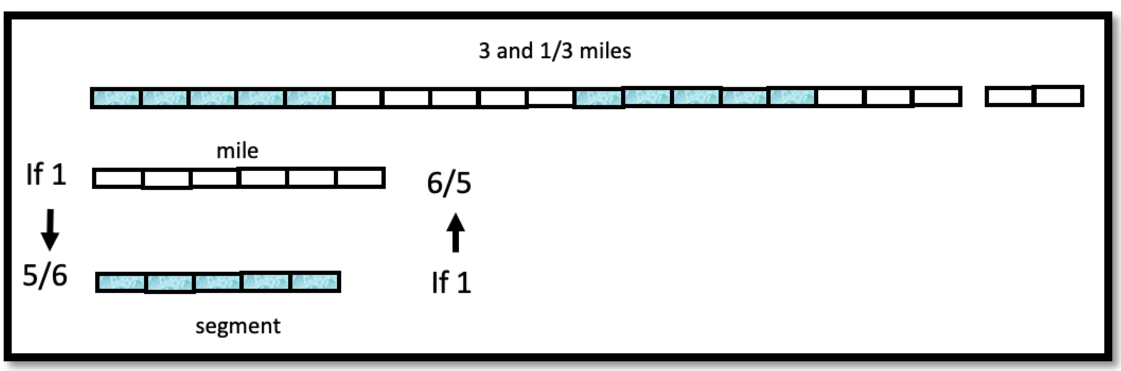

Figure 7.9. From mile as a unit to its part as a unit.

The situation is more difficult in the case of division used as an operation needed to find a missing factor when another factor and the product are given. For example, when twice the distance (d) from A to B is 10 miles, we have the equation $2d = 10$ whence $d = 10 \div 2 = 5$. This time, just as in the case of building four-tile towers out of eight tiles, the meaning of the division operation used to find *d* is distributing 10 miles within 2 distances (this distribution is the partition model for division). The result is 5 miles for the distance which is the unit. Now, replacing division by multiplication and inverting 2 yield $10 \cdot \frac{1}{2} = \frac{1}{2} \cdot 10 = 5$. That is, the distance *d* became $\frac{1}{2}$ of 10 miles. Note that taking $\frac{1}{2}$ of 10 is the same as dividing 10 by 2. That is, once again, in the case of partition model for division the *Invert and Multiply* rule can be understood in terms of the dividend-divisor context.

Consider now a situation when 5/9 of unknown distance *d* is equal to 20/3 miles. Finding *d* from the equation $\frac{5}{9} d = \frac{20}{3}$, leads to the division operation $d = \frac{20}{3} \div \frac{5}{9}$. This time, the process of distributing $\frac{20}{3}$ miles within $\frac{5}{9}$ distances (*d*) is difficult to imagine for it is an abstraction; yet the distance *d* may be the unit when the operation $\frac{20}{3} \div \frac{5}{9}$ is considered. Replacing $\frac{20}{3} \div \frac{5}{9}$ by $\frac{20}{3} \cdot \frac{9}{5}$ and the latter by $\frac{9}{5} \cdot \frac{20}{3}$ means the change of unit; that is, the distance *d* becomes $\frac{9}{5}$ of $\frac{20}{3}$ miles, where $\frac{20}{3}$ (miles) is the new unit. The product $\frac{9}{5} \cdot \frac{20}{3}$ means taking $\frac{9}{5}$ of $\frac{20}{3}$ miles. In that way, $d = \frac{9}{5} \cdot \frac{20}{3} = 12$ (miles). The meaning of operations in the case of fractions with a missing factor can only be explained by referring to a similar case that involves whole numbers; otherwise, the partition model, unlike the measurement model, is a pure abstraction. That is, depending on context, either partition or measurement model may be seen as an abstraction (cf. the case discussed in section 7.5.1).

7.6.3 The Invert and Multiply rule as a change of unit

Where can one recognize in the *Invert and Multiply* rule the change of unit when making a transition from $\frac{3}{4} \div \frac{1}{3}$ to $\frac{3}{4} \cdot 3$ accoring to this rule? Which one is the original unit (in the case of division) and which one is the new unit (in the case of multiplication)? To answer these questions, let $\frac{3}{4} \div \frac{1}{3} = x$. Then $\frac{1}{3}x = \frac{3}{4}$. The first fraction in the last relation is $\frac{1}{3}$. One may assume that x is the original unit in the problem with a missing factor. This unit, x, and its one-third are shown in Figure 7.10, parts (i) and (ii), respectively. Therefore, the new unit is the region for which $\frac{1}{3}x$ is equal to $\frac{3}{4}$. This new unit is shown in Figure 7.10 (iii), the non-shaded section of which is $\frac{1}{4}$ and the shaded part is $\frac{3}{4}$. Then, the found value of x, $x = 3 \cdot \frac{3}{4} = \frac{9}{4}$, is shown in Figure 7.10 (iv).

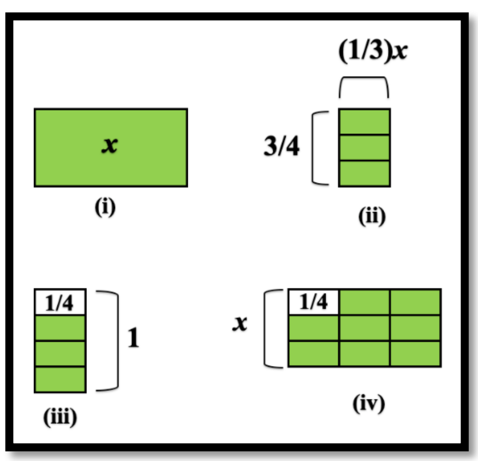

Figure 7.10. From x being a unit to $4x/9$ being a unit.

7.7 Egyptian fractions

History of mathematics can provide interesting mathematical ideas and, according to the Conference Board of the Mathematical Sciences (2012), is worth to "be woven into existing mathematics courses" (p. 61). A history of ancient Egyptian mathematics (around 1650 BC) provides such context for both conceptual and procedural mathematical learning. An Egyptian fraction is a sum of the finite number of distinct unit fractions, for example, $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$ is such a fraction. A unit fraction itself, like $\frac{1}{3}$, is an Egyptian fraction, although it can be represented as a sum of other unit fractions, like $\frac{1}{3}=\frac{1}{4}+\frac{1}{13}+\frac{1}{156}$. Sometimes, an Egyptian fraction is understood as a special representation of a fraction (alternatively, a positive rational number) through a finite sum of distinct unit fractions, so that the sum $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$ which is equal to $\frac{71}{105}$ is not considered an Egyptian fraction representation of $\frac{71}{105}$. Indeed, if one asks *Wolfram Alpha* to represent $\frac{71}{105}$ as an Egyptian fraction, the answer is $\frac{1}{2}+\frac{1}{6}+\frac{1}{105}$. However, the term Egyptian fraction can be used for any finite sum of distinct unit fractions.

Often, Egyptian fractions can be used to solve simple division word problems more effectively in comparison with the use of the dividend-divisor context (section 7.1), such as dividing 5 (identical) pies among 6 people as shown in Figure 7.11. The equality $\frac{5}{6}=\frac{1}{2}+\frac{1}{3}$, the right-hand side of which is an Egyptian fraction, can be used to decide the division in a different way (Figure 7.12): 3 pies are divided into 2 equal pieces each and 2 pies are divided into 3 equal pieces each (rather than each pie in 6 pieces as in Figure 7.11). Consequently, each person would get half of a pie plus one-third of it. The total number of pieces in Figure 7.11 is 30 and the total number of pieces in Figure 7.12 is 12. Another example follows from the equality $\frac{3}{5}=\frac{1}{2}+\frac{1}{10}$ which can be used to decide the division of 3 pies among 5 people. All the pies are divided in half and then one of the halves is divided into 5 equal parts so that each person would get half of a pie plus one-tenth of a pie. Here, we have 10 pieces of pie rather than 15 when using the dividend-divisor context for fractions (see Figure 7.1). These examples should not be taken to mean that an Egyptian fraction always provides fewer pieces than the dividend-divisor context. For example, dividing 2 pies among 5 people through the dividend-divisor context yields 10 equal pieces. At the same time, $\frac{2}{5}=\frac{1}{3}+\frac{1}{15}$ is the Egyptian fraction representation of the same division which requires to divide each pie into 3 equal pieces and then divide one such piece into 5 equal pieces. As a result, once again, we have 10 pieces: 5 of them are 1/3 of a pie and another 5 are 1/15 of a pie.

A more complicated example is provided by the Egyptian fraction $\frac{3}{7}=\frac{1}{3}+\frac{1}{11}+\frac{1}{231}$. The dividend-divisor context yields 21 equal pieces by dividing each pie into seven equal pieces. The Egyptian fraction yields 7 pieces, each of which is 1/3 of a pie; then 7 pieces, each of which is 1/11 of a pie; and another 7 pieces each of which is 1/231 of a pie. Once again, both divisions yield 21 pieces. Note that $7\cdot(\frac{1}{3}+\frac{1}{11}+\frac{1}{231})=\frac{7}{3}+\frac{7}{11}+\frac{1}{33}=\frac{77+21+1}{33}=\frac{99}{33}=3$; that is, putting all the 21 pieces together yields 3 pies that were divided among seven people in a fair (but rather complicated) way.

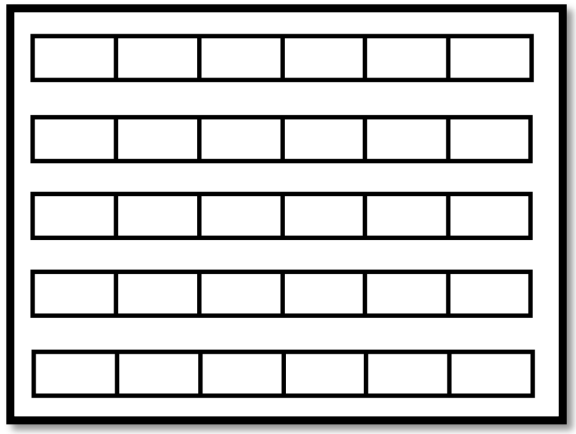

Figure 7.11. Dividing five pies among six people using dividend-divisor context.

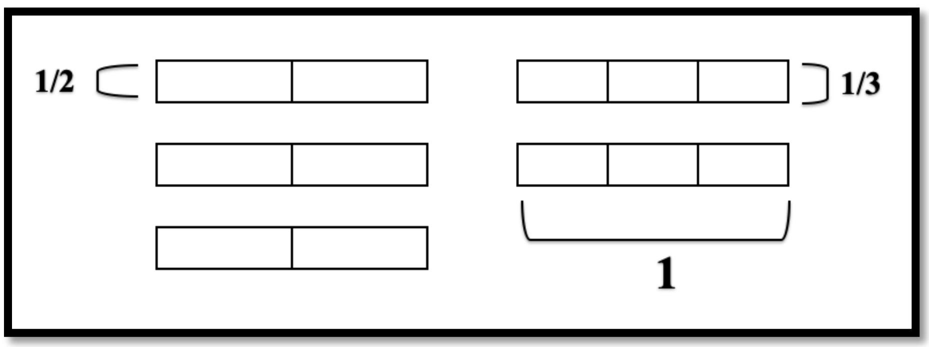

Figure 7.12. Dividing five pies among six people using the Egyptian fraction $\frac{5}{6}=\frac{1}{2}+\frac{1}{3}$.

CHAPTER 8: FROM FRACTIONS TO DECIMALS TO PERCENT

> "Because elementary teachers prepare their students for the middle grades, courses and seminars for elementary teachers should also attend to how the mathematical ideas of elementary grades build to those at the middle grades" (Conference Board of the Mathematical Sciences, 2012, p. 25).

8.1 From long division to fractions as decimals

The algorithm of long division (discussed in Chapter 2) can be used to demonstrate where the decimal representation of a common fraction comes from. The algorithm allows on to replace hidden in the common fraction division by a number written in a decimal notation the digits of which represent the number of 1/10's, 1/100's, 1/1000's, and so on included in the fraction. For example, the common fraction 5/16 can be transformed into a decimal fraction by dividing 16 into 5 through long division (Figure 8.1). The result is the number 0.3125 where the digits 3, 1, 2, and 5 represent, respectively, the number of 1/10's, 1/100's, 1/1000's, and 1/10000's included in 5/16; that is,

$$\frac{5}{16} = 0.3125 = \frac{3}{10} + \frac{1}{100} + \frac{2}{1000} + \frac{5}{10000} = 3 \cdot 10^{-1} + 1 \cdot 10^{-2} + 2 \cdot 10^{-3} + 5 \cdot 10^{-4}.$$

This representation is similar to representing the integer 3125 as $3 \cdot 10^3 + 1 \cdot 10^2 + 2 \cdot 10^1 + 5 \cdot 10^0$. The right-hand side of Figure 8.1 shows also the face values 3, 1, 2, and 5 as quotients of four divisions until the zero remainder is reached and thus, if division continues, the remaining quotients would also be zeros. This means that the decimal representation of 5/16 is a terminating decimal. At the same time, 2/5 = 0.4 and, therefore, zero remainder is reached after the first division (of 20 by 5). In particular, one can see that in base-ten arithmetic, not only "non-negative integers represented in base ten can be viewed as "polynomials in 10"" (Conference Board of the Mathematical Sciences, 2012, p. 59), but common fractions can also be viewed as polynomials in the negative integer powers of ten.

Another observation that one can make from decimal representations of common fractions is that numeric comparison of 0.3125 and 0.4 is easier than that of 5/16 and 2/5. This is precisely because, while common fractions can be seen representing the dividend-divisor context which extends the division of integers to the domain of fractions when the dividend is not a multiple of the divisor, their decimal equivalents represent notations with completed division. The right-hand side of Figure 8.1 shows an alternative representation of the long division through the sequence of relations among dividend, divisor, quotient, and remainder (DDQR), in which the quotients are the face values in the decimal representation of the common fraction 5/16.

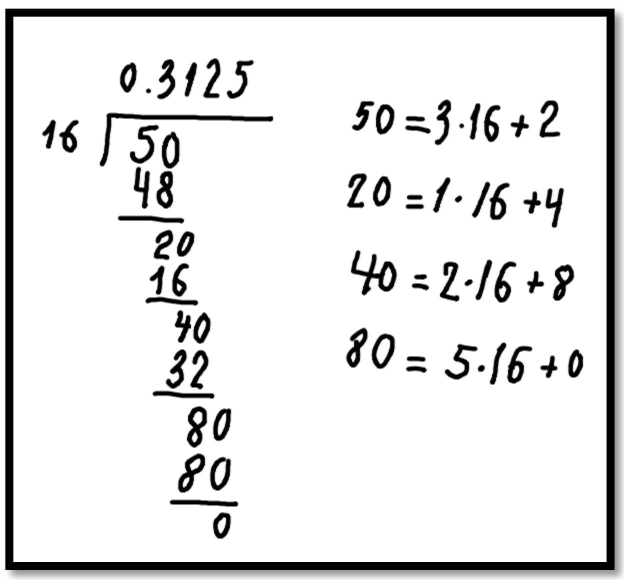

Figure 8.1. Long division of 5 by 16 with 4 different remainders: 2, 4, 8, and 0.

The process of long division can demonstrate that, for some fractions, zero remainder may never be reached. In that case, long division never terminates; however, the behavior of the face values it produces is periodic. As an example, consider the process of obtaining the decimal representation of the fraction 3/7 through long division. As shown in Figure 8.2, the long division goes through the sequence of six DDQR relations until such relations begin repeating. That is, this repetition is due to a simple yet conceptually profound fact that when an integer smaller than 7 (e.g., 3) is divided by 7, the corresponding remainder may not be greater than or equal to 7. Indeed, only six DDQR relations are possible:

$$1 = 0 \cdot 7 + 1,\ \ 2 = 0 \cdot 7 + 2,\ \ 3 = 0 \cdot 7 + 3, 4 = 0 \cdot 7 + 4, 5 = 0 \cdot 7 + 5, 6 = 0 \cdot 7 + 6.$$

Therefore, 3/7 = 0.428571428571428571... . Such non-terminating periodic decimal has a special notation: $3/7 = 0.\overline{428571}$ with the repeating part of the decimal having a bar above it. Likewise, $1/3 = 0.3333... = 0.\overline{3}$, $2/3 = 0.6666... = 0.\overline{6}$, and $1/11 = 0.090909... = 0.\overline{09}$.

In general, when an integer smaller than *n* is divided by *n*, the *largest* number of different remainders is equal to *n* – 1. The corresponding quotients become face values and they repeat each other by forming cycles the length of which may not be greater than *n* – 1. In the case when the prime factorization of *n* consists of the powers of two and five only, the common fraction with the denominator *n* is represented by a terminating decimal; otherwise, the fraction is represented by a non-terminating decimal the face values of which form cycles of length not greater than *n* – 1. In other words, through the process of long division by 7, students and others may see that the process is repeating the same calculations over and over, and can recognize that the corresponding fraction is represented by a repeating decimal. Through developing such representation, one can understand what it means to "express regularity in repeated reasoning" (Common Core State Standards, 2010, p. 8).

One can also use *Wolfram Alpha* to see that the periods of cycles in the decimal representations of the unit fractions with prime number (i.e., an integer divisible by one and itself only) denominators, say, 1/23, 1/31, 1/37, and 1/41 are, respectively, 22, 15, 3, and 5. However, hardly any pattern can be seen here. This demonstrates the complexity of mathematics behind the issue of relating the length of a cycle in the decimal representation of $1/n$ to n; something that is beyond the scope of this course.

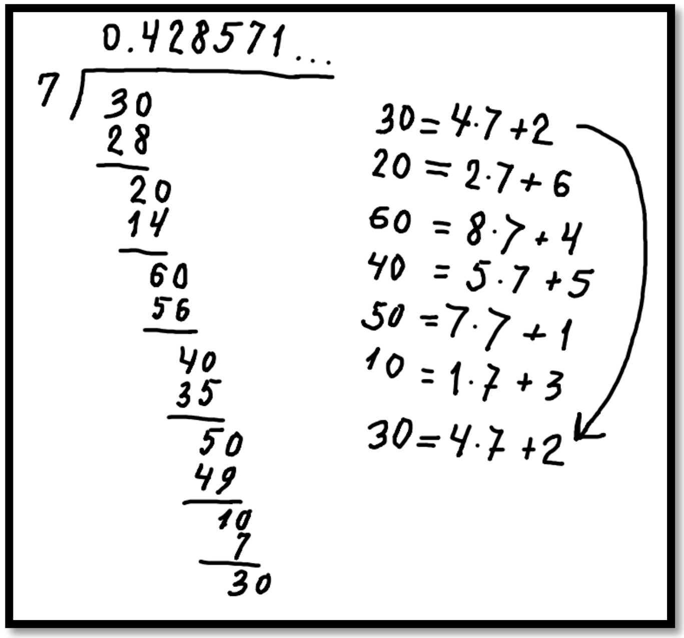

Figure 8.2. Long division by 7 with 6 different (non-zero) remainders, 2, 6, 4, 5, 1, 3.

8.2 From alternative comparison of fractions to percent

A rational number, two representations of which are fraction (including Egyptian fraction) and decimal, has another representation studied at the elementary level. This third representation is called percent. The term is well known to elementary school students from a number of sources. Many students even refer to certain conditions in terms of percent when talking about weather; e.g., fifty percent chance of snow (or rain). But just as the concept of chance (see Chapter 11), the concept of percent has to be formally introduced before it can become an alternative representation of a fraction. One way to introduce percent is through the comparison of fractions using a 100-cell grid. However, the effectiveness of comparing fractions on a 100-cell grid depends on their denominators, even if they are small integers. As an example, consider the fractions 3/5 and 7/10 – which one is bigger? To answer this question, both fractions can be represented as parts of a 100-cell grid shaped in the form of square. Because 1/5 and 1/10 of such grid cover, respectively, 20 and 10 cells, the fraction 3/5 covers 60 cells and the fraction 7/10 covers 70 cells (Figure 8.3). Therefore, 7/10 > 3/5.

Consequently, one can say that by representing 3/5 on a 100-cell grid, 60 cells per 100 cells were shaded; or, using the Latin word *centum* (translated as *hundred*), 60 percent of a 100-cell grid are shaded. The word percent has the following (well-known) notation: %. Therefore, 3/5 = 60%. Likewise, 7/10 = 70%. Alternatively, 3/5 = 60/100 = 60% and 7/10 = 70/100 = 70%.

Note that the denominators 5 and 10 can easily be multiplied to become 100; indeed multiplying 5 by 20 yields 10 and multiplying 10 by 10 yield 100 as well. In other words, the denominators of the fractions 3/5 and 7/10 are integer factors of 100. The denominators may not necessarily be integer factors of 100. For example, 3/40 = 1.5/20 = 7.5/100 = 7.5% and 7/80 = 35/400 = 8.75/100 = 8.75%.

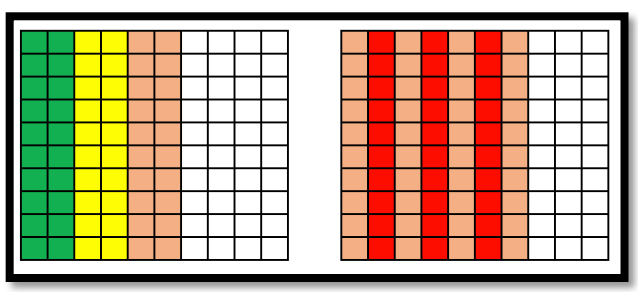

Figure 8.3. 3/5 = 60% and 7/10 = 70% .

Certain fractions, the denominators of which are not multiples of 2 and/or 5 can still be easily represented on a 100-cell grid. Consider the fraction 1/3. One can see (Figure 8.4) that 33 cells represent approximately 1/3 of a 100-cell grid with another 66 cells representing approximately 2/3 of the grid. The remaining cell, which is 1%, can be divided into three equal parts, each of which is 1/3%. Therefore, $\frac{1}{3} == 33\frac{1}{3}\% = 33.\overline{3}\%$ and $\frac{2}{3} == 66\frac{2}{3}\% = 66.\overline{6}\%$. One can see that $\frac{1}{3} = \frac{100}{3}\% = 33\frac{1}{3}\%$ and $\frac{2}{3} = 2 \cdot \frac{1}{3} = 2 \cdot \frac{100}{3}\% = \frac{200}{3}\% = 66\frac{2}{3}\%$. That is,

$\frac{a}{b} = a \cdot \frac{1}{b} = a \cdot \frac{100}{b}\% = \frac{100a}{b}\%$.

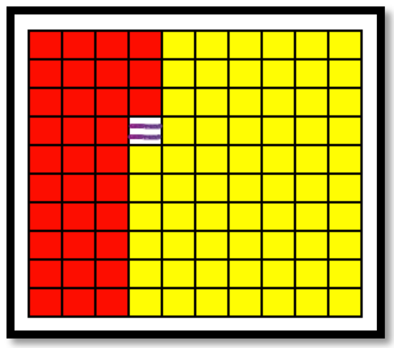

Figure 8.4. 1/3 and 2/3 as percent.

However, unlike 1/3, the decimal representation of which is $0.\overline{3}$, the fraction $1/7 = 0.\overline{142857}$ cannot be effectively represented on a 100-cell grid. Indeed, because $\frac{1}{7} = \frac{100}{7}\% = 14\frac{2}{7}\%$, the fractional part of the last (percentage) number, i.e., $\frac{2}{7}\%$, does not have a visually lucid representation on a 100-cell grid in comparison with $\frac{1}{3}\%$ (which is just 1/3 of a cell). This, however, does not mean that percentage representation of the fraction 1/7 is difficult to obtain: keeping in mind that 1 = 100%, the difference is in the notation used as the unit, 1 vs. 100%, the fractions of which have to be computed.

CHAPTER 9: RATIO AS A TOOL OF COMPARISON OF TWO QUANTITIES

> "Through their learning in the Ratios and Proportional Relationships domain, students [in Grade 6] ... understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities" (New York State Education Department, 2019, pp. 77, 78).

9.1. Different definitions of ratio

What is a ratio? Often a ratio of two numbers is defined as a characteristic which shows how many times one number contains another number; in other words, a ratio is defined as a result of measuring one quantity by another quantity. For example, one can measure one side of a rectangle by another (adjacent) side of the rectangle. In the case when the larger side length of a rectangle is twice as large as its smaller side length and when the former is measured by the latter, one can say that the ratio of the side lengths is two to one. A ratio is a number and in that case the ratio is equal to the number 2. One can also measure the smaller side length by the larger side length and say that the ratio is one to two. Just as in the case of using measurement context to introduce a fraction, the ratio in the latter case is the number 1/2. That is, as a common fraction, ratio is the result of dividing one number by another number when the larger number is considered as the whole and the smaller number is considered as its part. One can recognize the part-whole context for a ratio here because the whole is divided in two equal parts, one of which is 1/2 of the whole. According to Common Core State Standards (2010), students in Grade 6 "expand the scope of problems for which they can use multiplication and division to solve problems, and they connect ratios and fractions" (p. 39). A classic example of a ratio is the so-called Golden Ratio, $\frac{1+\sqrt{5}}{2}$, when, for example, in a regular pentagon (Figure 9.1) one measures its diagonal (AC) by its side

(BC) and concludes (although in abstract form) that the diagonal, being longer than the side, includes the latter $\frac{1+\sqrt{5}}{2} \approx 1.61803$ times.

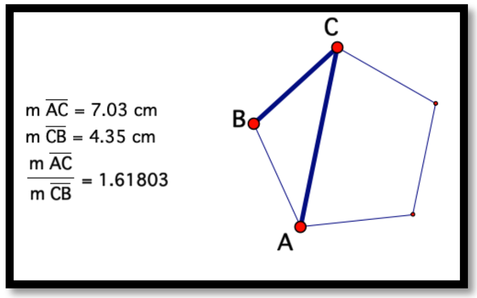

Figure 9.1. AC:BC = $\frac{1+\sqrt{5}}{2} \approx 1.61803$.

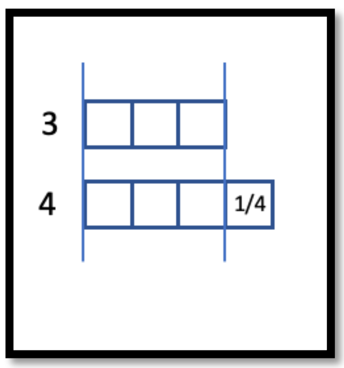

Figure 9.2. Measuring 3 by 4 yields 3/4 as the ratio 3 to 4.

Understanding of how to measure one number by another number (i.e., what it means numerically for one number to include another number) depends on an interpretation of this measurement (inclusion). For example, one can interpret the process of measuring the number 3 by the number 4 as follows. Let us measure a pie cut into three pieces by a larger pie cut into four pieces, assuming that all pieces are identical as shown in Figure 9.2. When 3 is measured by 4, this operation can be put in a pictorial context from which it follows that 3 is 3/4 of 4. That is, 4 as a measuring tool is comprised of four equals parts, and 3 as an object of measurement is comprised of three such parts. One of the parts that form 4 is 1/4 of 4; therefore, measuring 3 by 4 yields numerically 3/4 of 4. In other words, the ratio of 3 to 4 is 3/4 as the result of measuring 3 by 4. Likewise, measuring 5 by 7 yields the ratio 5/7, measuring 6 by 11 yields the ratio 6/11, and so on.

Division of two numbers can be introduced in the partition context also. So is the ratio and its interpretation. For example, it is often said that the ratio of students to computers in a school is, say, two to one. This statement can be interpreted as the case when students are partitioned among computers; for instance, 10 students can be partitioned among 5 computers yielding two students for a computer (just as 10 apples can be partitioned among 5 people yielding 2 apples for a person). The same numeric outcome (i.e., ratio of students to computers) would be in the case of 100 students and 50 computers, or 200 students and 100 computers. In the reciprocal relation, the ratio of computers to students in all mentioned cases is one to two; that is, the reciprocal ratio is equal to the fraction 1/2 (alternatively, 0.5 or 50%).

9.2. Introducing ratio as a tool

Whatever a context, a ratio can be introduced as a tool of comparing two quantities. This tool is the quotient which, when expressed as a fraction, is written in the simplest form or when the latter form is described in terms of the dividend-divisor context for a fraction (see Chapter 7). That is, the ratio 10 to 3 is the fraction 10/3 understood as the number 3 divided into the number 10. Likewise, the ratio 3 to 10 is understood as the fraction 3/10 which, in turn, can be interpreted as 3 identical objects (e.g., pizza or cake) divided into 10 equal parts. Obviously, 3 computers cannot be divided into 10 parts to allow each student to have 3/10 of a computer to use. But the ratio shows a possible application of the dividend-divisor context for fractions when two quantities have to be compared. One may recall that a difference may be used as a tool to compare quantities. For example, the pairs of numbers (5, 2), (6, 3) and (10, 7) are in the same difference relation. Indeed, 5 – 2 = 6 – 3 = 10 – 7 = 3. A context for using difference as a tool of comparison may be as follows. When among 5, 6 and 10 apples there are, respectively, 2, 3 and 7 red apples, each collection of apples have the same number of apples that are not red. And here 2 is a part of 5, 3 is a part of 6, and 7 is a part of 10. But there may be other pairs of numbers, like (5, 2), (10, 4) and (15, 6), which form equal fractions: 5/2, 10/4 and 15/6 (or 2/5, 4/10 and 6/15). Yet, contexts for those pairs are different and the ratio is more concrete when 2 is *not* a part of 5, 4 is *not* a part of 10 and 6 is *not* a part of 15. Indeed, the fraction 5/2 when interpreted as the result of dividing 5 by 2 (or 2 divided into 5) can describe baskets with 5 oranges and 2 apples, 10 oranges and 4 apples, 15 oranges and 6 apples, or 50 oranges and 20 apples. All such baskets are the same in terms of having oranges to apples ratio equal to the fraction 5/2 or the decimal 2.5 or the percent 250%. Likewise, the reciprocal ratio can be written as 2/5, 0.4, or 40%.

9.3 Using ratio to find an unknown quantity

Consider

Problem 1. *The ratio of dogs to cats in a household is 1 to 2. If there are three dogs in the household, how many cats are there*?

To solve this problem, one can begin with the smallest household in which the ratio of dogs to cats is 1 to 2. Figure 9.3 shows one dog and two cats. If one measures the frame which includes a dog with the frames which include cats, the result of this measurement is 1/2. When there are three dogs in a household (Figure 9.4), then, in order to preserve the 1 to 2 ratio each dog would bring two cats; thus, for three dogs (a new unit) there are six cats (two new units).

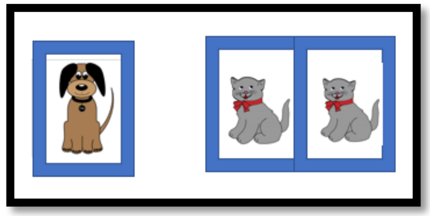

Figure 9.3. The smallest household with the dog to cat ratio equal 1 to 2.

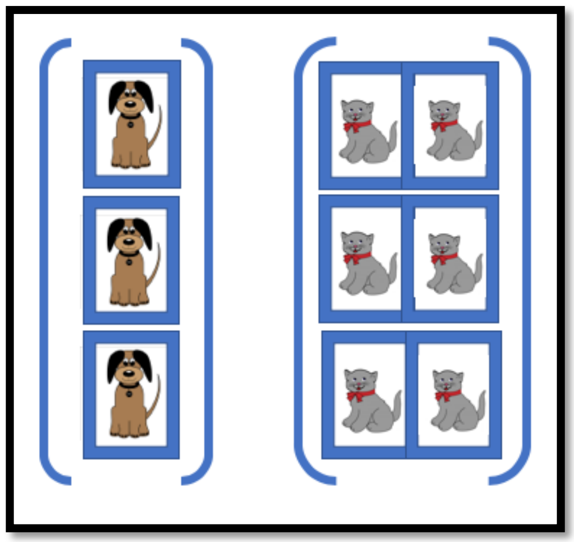

Figure 9.4. Each dog brings two cats to preserve the 1 to 2 ratio.

On a formal level, there is an unknown number of cats, *x*, and the known number of dogs, 3. The ratio of 3 dogs to *x* cats is the fraction 3/*x*. But this ratio is equal to 1/2. The two ratios may be equated in the form called proportion $\frac{3}{x} = \frac{1}{2}$, an equation between two ratios. To find *x* from this equation, both ratios (i.e., fractions) can be written with the same denominators, $\frac{3 \cdot 2}{x \cdot 2} = \frac{1 \cdot x}{2 \cdot x}$, whence $x = 3 \cdot 2 = 6$. Alternatively, three dogs can be presented in the form of a single unit, that is, three dogs for a unit. Consequently, x cats have to be presented in the form of two units, $x/2$ cats for a unit. This results in the proportion $\frac{x}{2}=\frac{3}{1}$ whence $x = 3 \cdot 2 = 6$.

Problem 2. *The ratio of dogs to cats in a shelter is 3 to 4. If there are 12 dogs in the shelter, how many cats are there*?

Once again, we begin solving the problem with the smallest shelter with the dog to cat ratio equal 3 to 4. As shown in Figure 9.5, twelve dogs can be presented in the form of three units of four dogs in each. To preserve the ratio 3 to 4, there has to be four units with the total unknown number, x, of cats. That is, $\frac{12}{3}=\frac{x}{4}$ whence $x = 4 \cdot \frac{12}{3} = 16$. Here, the fraction 12/3 shows that 12 dogs were put in three groups (units) to create a new unit of four dogs. Therefore, one must have four units of cats with four cats in a unit.

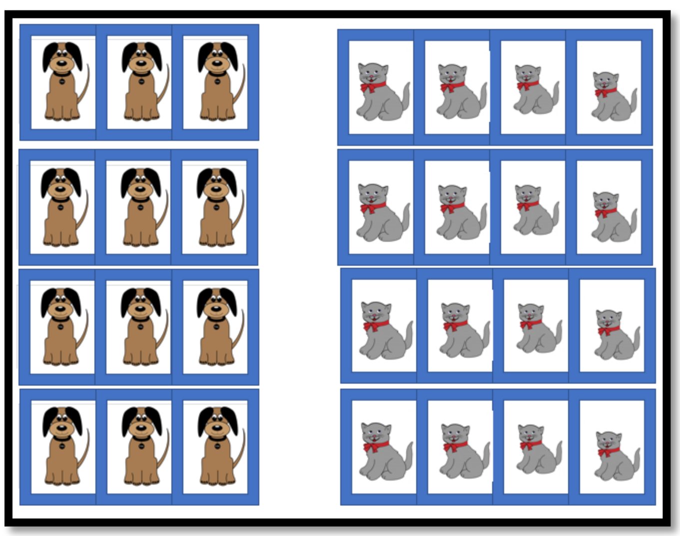

Figure 9.5. Each triple of dogs brings quadruple of cats.

Problem 3. The ratio of students to professors in a community college is 51 to 4. If there are 848 professors, how many students are there?

Obviously, one cannot support solving this problem by drawing a picture. We did use picture in the case of small numbers and, in doing so, attempted to develop conceptual understanding of a formal problem-solving strategy. In the case of large numbers, an algorithm of constructing a proportion between two ratios has to be used as an applicable strategy. If x is an unknown number of students in the college, then $\frac{x}{848} = \frac{51}{4}$ whence $x = \frac{51}{4} \cdot 848 = 10,812$.

9.4 Problems that require insight to avoid an error in using proportional reasoning

Figure 9.6 shows a 3-step high ladder in which each step requires three bars. One can say that the ratio of steps to bars is 1 to 3. The following question may be formulated: How many bars are needed for a 25-step high ladder. To answer this question, the proportion $\frac{3}{1} = \frac{x}{25}$ has to be constructed whence $x = 75$.

One can see that the number of bars (b) and the number of steps (s) are in the (proportional) relation $b = 3 \cdot s$. That is, the number 3 is the bars to steps ratio. In general, it can be said that the variables x and y are in the same ratio if there exists number n such that $y = n \cdot x$. As a common extension of the ladder problem, consider the ladder shown in Figure 9.7. The question to be answered is: How many bars are needed for the 25-step high ladder. If one already knows that the 25-step ladder extension shown in Figure 9.6 is 75, then one has to add just one bar to 75 bars to have 76 as the answer. But this is an insightful solution. Sometimes, the ratio approach is applied to the ladder of Figure 9.7 noting that the ratio of steps to bars is 4 to 10. So, by constructing the proportion $\frac{10}{4} = \frac{x}{25}$, one ends up with x equal to 62.5. But x has to be whole number, unlike 62.5. An error is due to the fact that the number of steps and the number of bars are in the relation

$b = 3s + 1$ and, therefore, proportional reasoning may not be applied to the ladder of Figure 9.7, unless one solves the problem without the bottom bar and then add 1 to 75. As mentioned in the Standards for Preparing Teachers of Mathematics (Association of Mathematics Teacher Educators, 2017), "beginning teachers [should] be aware of likely misconceptions, and be open to understanding unique ways that students might use to express characteristics and generalizations" (p. 86). Likewise, Conference Board of the Mathematical Sciences (2012) recommends that teacher candidates "need the ability to find flows in students' arguments and to help their students understand the nature of errors" (p. 2).

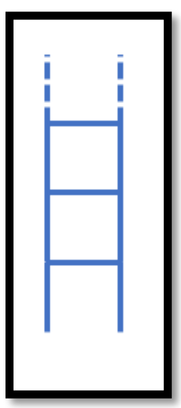

Figure 9.6. Ladder without the bottom rod.

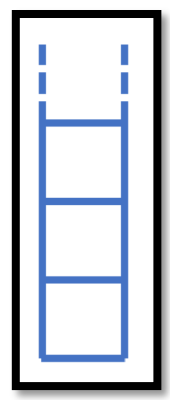

Figure 9.7. Ladder with the bottom rod.

For more information on the erroneous use of proportional reasoning see (Abramovich and Brouwer, 2011).

CHAPTER 10: GEOMETRY

> "Certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said mechanical method did not furnish an actual demonstration. But it is of course easier, when the method has previously given us some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge" (Archimedes, 1912, p. 13)[2].

10.1 Geoboard activities and Pick's formula.

A geoboard is a hands-on learning environment for exploring basic geometric ideas associated with different polygonal shapes. The environment allows for the construction of a variety of polygons by using rubber bands held by pegs. A polygon on a geoboard can be associated with the number of pegs that the rubber band touches. Another characteristic of a polygon on a geoboard is the number of pegs in its interior. Thus, polygons may be compared in terms of the numbers of two types of pegs related to each of them. For example, in one case, a rubber band touches six pegs and encloses three pegs; in another case, a rubber band touches eight pegs and encloses two pegs. As will be shown below, the two polygons have equal areas. *In a mean time, without knowing how to find area, one can explore whether on a geoboard both polygons can be triangles*.

This association of polygons, both convex and concave, with pegs on a geoboard brings about counting as one of the major problem-solving strategies furnishing geoboard geometry with an informal flavor, something that is especially important at the elementary level. At the same time, appropriately designed counting activities in the context of geoboard can be conceptually rich and the exploration suggested at the end of the last paragraph is such an example. Furthermore, because on a geoboard a linear unit is a side of a unit square the vertices of which are the four pegs closest

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[2] Archimedes—a Greek mathematician of the 3rd century BC.

to each other, one can find area of any shape by using a strategy shown in Figure 10.1 – enclose the shaded shape into a rectangle (square) and then subtract from its area the (easy to find) areas of extraneous triangles as they always have half of area of a rectangle which, in turn, is comprised of unit squares. This is due to the fact, which can be confirmed through informal geometry (see section 10.3 below) using paper and scissors, that a diagonal of a rectangle cuts it in two congruent triangles. In other words, addition and subtraction are two major arithmetical operations in the context of finding areas on a geoboard. Furthermore, the areas on a geoboard are always multiples of one-half of area of the unit square; in other words, area on a geoboard is always an integer divided by two. Expectations of Common Core State Standards (2010) for students in Grade 6 include finding "areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging of removing pieces, and relating the shapes to rectangles" (p. 40).

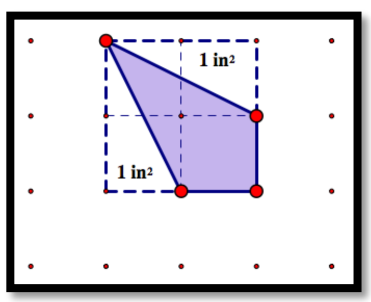

Figure 10.1. Finding area of the shaded polygon using informal geometry.

Consider seven polygons shown in Figure 10.2. One can see that the first four polygons have their areas increased by 1/2 as the number of their border pegs increases by one and there is no increase in the number of internal pegs. Indeed, using the strategy shown in Figure 10.1 yields

$$A_1 = 4-(1+1+\frac{1}{2}) = \frac{3}{2},\ A_2 = 4-(1+1) = \frac{4}{2},\ A_3 = 4-(1+\frac{1}{2}) = \frac{5}{2},\ A_4 = 4-(\frac{1}{2}+\frac{1}{2}) = \frac{6}{2}.$$

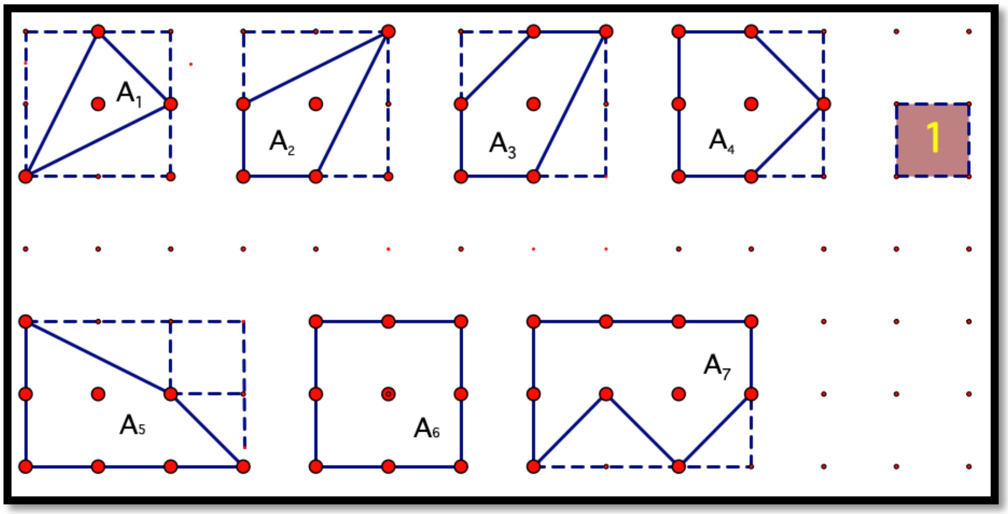

Figure 10.2. Finding areas of the polygons with $I = 1$ and $B = 3, 4, \ldots, 9$.

The same can be said about the three polygons of the bottom part of Figure 10.2. Indeed,

$$A_5 = 6 - (1 + 1 + \frac{1}{2}) = \frac{7}{2}, A_6 = 4 = \frac{8}{2}, A_7 = 6 - (\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) = \frac{9}{2}.$$

One can see that the numerator in a fraction representing area of each of the seven polygons is equal to the number of pegs (B) on its borders, respectively, and all the polygons have a single internal peg (I) the presence of which does not affect the area. From this observation, one can conjecture that when $I = 1$, area of a polygon with B border pegs and one internal peg, $A(B, 1)$, can be computed through the formula $A(B, 1) = B/2$.

Figure 10.3 shows a different situation. The first four polygons with four border pegs ($B = 4$) and one internal peg ($I = 1$) have $A(\mathrm{B}, 1) = 4/2 = 2$ as they are similar to those shown in Figure 10.2. The next group of four polygons with four border pegs ($B = 4$) and two internal pegs ($I = 2$) have area $A(B, 2) = 3 = B/2 + 1$. The last group of four polygons with four border pegs ($B = 4$) and three internal pegs ($I = 3$) have area $A(B, 3) = 4 = B/2 + 2$. One can conjecture that the second term in the formulas for area is one smaller than the number of internal pegs. Thus, one can compete the conjecture by writing down the following formula (known as Pick's formula[3]) for area of a polygon on a geoboard

$$A(B, I) = \frac{B}{2} + I - 1 \ . \tag{10.1}$$

Note that a formal proof of Pick's formula, which was just conjectured above based on a number of cases in favor of formula, is complex and beyond the scope of the elementary school mathematics curriculum.

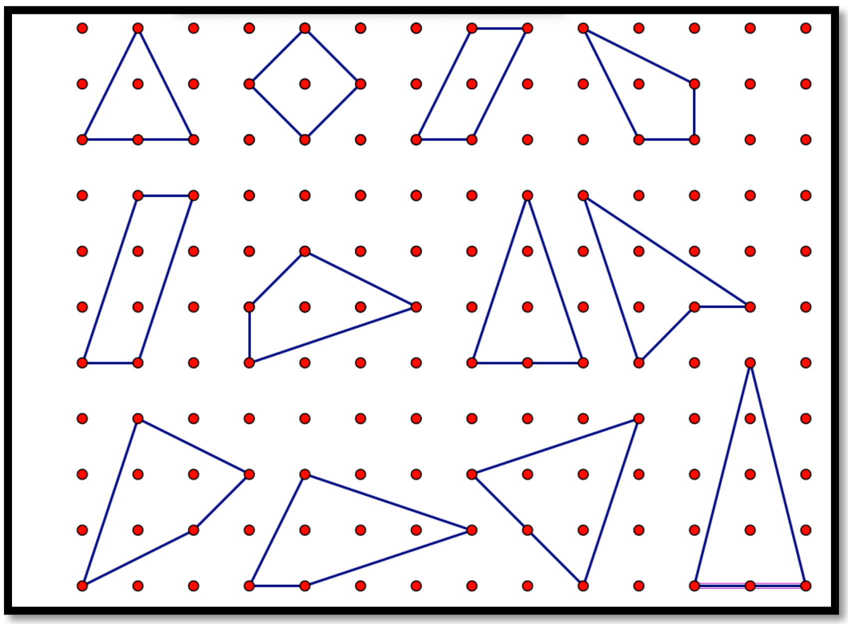

Figure 10.3. Finding areas of the polygons with $B = 4$ and I=1,2, 3.

Using Pick's formula, one can integrate computational fluency with conceptual understanding. Consider the polygon pictured in Figure 10.4. According to formula (10.1), its area is six square units. A conceptual part of dealing with Pick's formula is to explore ways of

3 George Alexander Pick (1859-1942), an Austrian mathematician who died at the age of 83 in the Nazi concentration camp.

modifying the polygon to have area smaller/greater than (or even equal to) six. Figure 10.5 shows how the polygon can be modified to have area seven square units. This modification does not change the number of internal pegs but, instead, it increases the number of border pegs by two. Figure 10.6 shows another modification of the polygon of Figure 10.4 by extending its size through replacing one border peg by another border peg and turning the replaced border peg into a new internal peg. This combination of the conceptual and the procedural supports the single question – multiple answers pedagogical philosophy. Thus, indeed, "geometry learning provides opportunities to develop ability to reason mathematically" (Association of Mathematics Teacher Educators, 2017, p. 52).

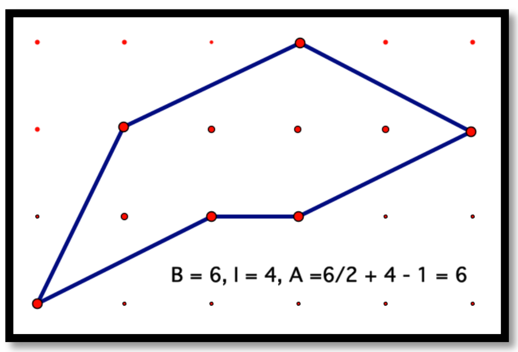

Figure 10.4 Using Pick's formula to find area.

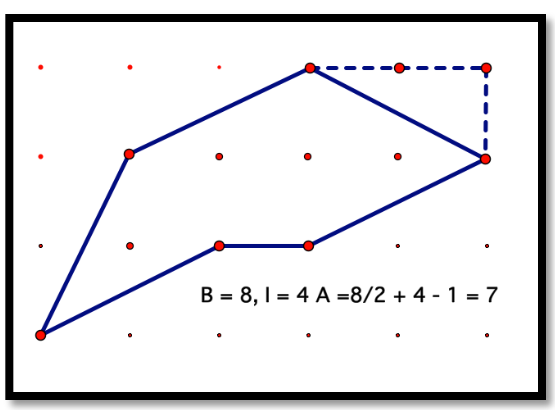

Figure 10.5. Modifying polygon of Figure 10.4 to have area 7.

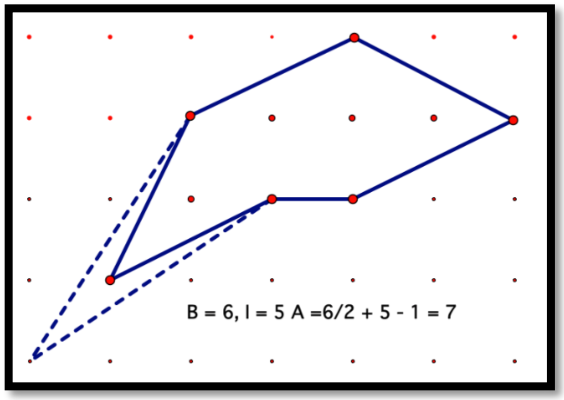

Figure 10.6. Another was of modifying polygon of Figure 10.4 to have area 7.

10.2 Tessellation.

A Manipulative Task (New York State Education Department, 1998). Using pattern blocks such as green (equilateral) triangles, blue rhombuses, red (isosceles) trapezoids, and yellow (regular) hexagons, have students discover the quantity of triangles needed to cover the blue rhombus, the red trapezoid, and the yellow hexagon.

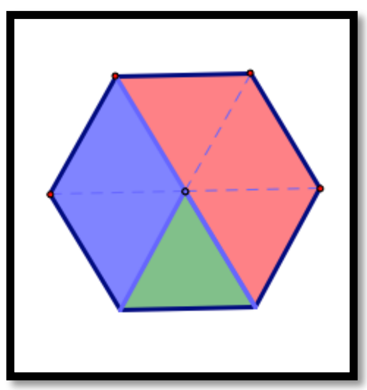

Figure 10.7. Hexagon built from triangle, rhombus, and trapezoid.

Recommended as a geometric activity appropriate for Grades 1 and 2, this manipulative task has a hidden meaning. It enables rather sophisticated mathematical concepts (that teachers need to know) to be gradually developed. In that way, the task can motivate various mathematical activities. For example, Figure 10.7 may be interpreted as covering the space around a point with no gaps or overlaps using six identical equilateral triangles (Figure 10.8), as the rhombus includes two triangles and the trapezoid includes three triangles. In arithmetical terms, we have the following relations: $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$, $\frac{1}{2}=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}$, $\frac{1}{3}=\frac{1}{6}+\frac{1}{6}$. The geometric construction shown in Figure 10.8 can then be extended to the whole plane by covering, step by step, the space around any point with the triangles. In much the same way, one can cover the whole plane with trapezoids (not necessarily isosceles; this remark will be explained later) and rhombuses. Such geometric activities are commonly referred to as tessellations. A few definitions can be introduced (for more details see (Abramovich, 2010)).

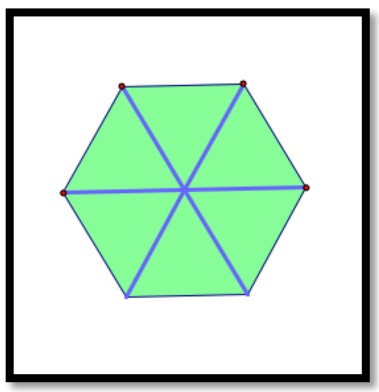

Figure 10.8. Hexagon built of six equilateral triangles.

Definition 1. A regular polygon is a polygon all sides of which are congruent and all angles of which are congruent.

Remark. Although a rhombus has four congruent sides, its angles are not congruent. Thus, only square is a regular quadrilateral.

Definition 2. A *tessellation* of the plane is a collection of plane figures that fills the plane with no gaps or overlaps.

Definition 3. A *regular tessellation* is a tessellation made up of congruent regular polygons.

One can say that Figure 10.7 represents a fragment of a regular tessellation with equilateral triangles.

Definition 4. A *semi-regular tessellation* is a tessellation with different regular *n*-gons in which the arrangement of polygons at every vertex is identical.

For example, Figure 10.7 shows a fragment of tessellation with three different polygons mentioned in the above manipulative task. This tessellation is not a semi-regular one because the rhombus and trapezoid are not regular polygons. An example of semi-regular tessellation is shown in Figure 10.9 in which square, regular hexagon and regular dodecagon (a polygon with 12 sides) cover space around a point with no gaps or overlaps. Note that $\frac{1}{4}+\frac{1}{6}+\frac{1}{12}=\frac{1}{2}$.

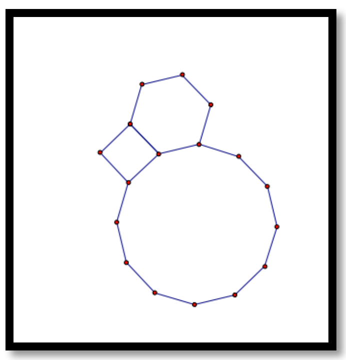

Figure 10.9. A semi-regular tessellation with square, hexagon, and dodecagon.

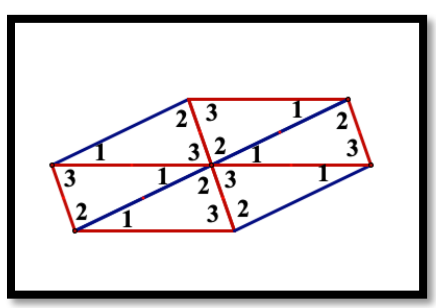

Figure 10.10. Tessellations with scalene triangles.

As was shown in Figure 10.8, by using six identical triangles one can cover space around a point without gaps or overlaps. The condition that triangles are identical is a critical one as equilateral triangles of different size (that is, similar equilateral triangles) would not provide the phenomenon of tessellation. An interesting activity is to explore if the condition of triangles being equilateral is needed for tessellation. To this end, a class may be asked to develop a set of six identical scalene triangles by giving each student a piece of paper, a straightedge, a pencil, and scissors. By folding paper into three equal parts (as if sending a letter in a business envelope) and then folding the so folded paper in half, one has six layers of paper. The next step is to use the straightedge to draw a scalene triangle on the top of the folded paper, then numerate angles using numbers 1, 2, 3, and cut out the triangle from the folded paper to get six identical triangles. Each triangle should then have angles labeled as on the sample copy. Now the task is to position the six triangles around a point in such a way that there are no gaps or overlaps (like in Figure 10.8). This arrangement of six triangles is shown in Figure 10.10. In this case, the methodology of informal deduction (the second level of geometric thinking in the van Hiele model introduced below in section 10.3) can be applied to conclude that all triangles tessellate. One can also see that the sum of three angles from three different (yet identical) triangles measures to 180°. But these three angles also belong to a single triangle. That is, the sum of three angles in any triangle is 180°, another conclusion which is due to informal deduction.

A similar activity can be carried out with quadrilaterals. First, one can use four identical squares (or rectangles) to show an obvious outcome – four identical squares (or rectangles) can be used to cover a space around a point with no gaps or overlaps. But as shown in Figure 10.11, a set of four identical quadrilaterals covers space around a point as well. The condition of quadrilaterals being identical is critical for the continuation of tessellation by proceeding from other points. Once again, each student in a class can be given a piece of paper which has to be folded in half twice and, like in the case of triangles, one has to draw a quadrilateral on the top of the folded paper, label its angles with the numbers 1, 2, 3, 4, and then cut out four copies of identical quadrilaterals. Once each student in the class can make the arrangement of quadrilaterals shown in Figure 10.11, using the methodology of informal deduction (see below section 10.3), the following conclusion can be made: all quadrilaterals tessellate. Likewise, one can conclude that the sum of four angles in any quadrilateral measures 360º.

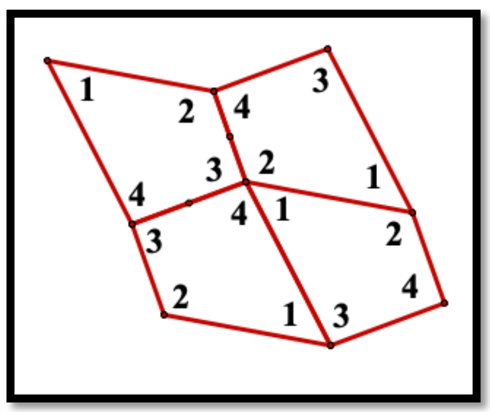

Figure 10.11. Tessellation with quadrilaterals.

10.3 The van Hiele levels of geometric thinking

Pierre van Hiele and Dina van Hiele-Geldof, Dutch mathematics educators, in the mid of the 20th century proposed a theory of how the learning of geometry occurs by students moving from one level of geometric thinking to another (higher) level. The theory suggests five levels, labeling them by the numbers 0, 1, 2, 3, and 4. Level 0, called visualization, is the basic level when students can only recognize geometric shapes by their appearance without making any conceptual analysis in order to put several seemingly different shapes in a single group. At Level 0, all shapes belong to the same group; young learners of mathematics do not see any difference between different quadrilaterals and sometimes even may not see a difference between circle and triangle. (Indeed, if sides of a tringle are not rigid, it can be transformed into a circle). Those are just shapes occupying space in the plane.

The next level is Level 1, called analysis, when students can classify polygonal shapes (triangles, quadrilaterals, pentagons) by the number of sides, by shapes being convex or concave, by shapes having curvilinear boundaries, and so on. In the primary grades, the main role of a teacher in teaching geometry is to help students to move from Level 0 to Level 1.

Level 2 is called informal deduction. It relates to an upper elementary level. A simple example of informal deduction is the use of paper and scissors in demonstrating that diagonal of rectangle divides it in two congruent triangles, because paper folding – another strategy of informal deduction that works in the case of demonstrating line symmetry – does not work in that case. A more complicated example of informal deduction is the derivation of Pick's formula (section 10.1) or the fact that all triangles and quadrilaterals tessellate (section 10.2).

Level 3, called formal deduction, is related to high school mathematics education. An example of that level of thinking is a formal proof that diagonal cuts rectangle in two congruent triangles by using the side-side-side property of the congruency of triangles. Put another way, all geometric properties taught in high school require the use of formal deduction.

Finally, the highest level of geometric thinking, Level 4, is called rigor. It relates to the studies of geometry by future mathematicians at the university level. At that level, students learn different axiomatic geometries. For example, they learn that all properties of geometric shapes in Euclidean geometry studied in high school follow from the axiom (a statement that belongs to the foundation of a certain theory and is taken as true without any proof) that through a point outside a straight line, one can draw only one straight line which is parallel to the first one. For many years, without success, mathematicians tried to prove this fact until it was assumed that more than one straight line can be drawn through a point and be parallel to the first line. This led to the creation of other geometries, different from Euclidean geometry.

10.4 On the relationship between perimeter and area of a rectangle

Conference Board of the Mathematical Sciences (2012) recommendations for elementary teacher candidates include the need to understand "the distinction and relationship between perimeter and area, such as by fixing a perimeter and finding the range of areas possible or by fixing an area and finding the range of perimeters possible" (p. 29). Towards this end, the following activities were designed in the context of rectangles.

Activity 1. Among all integer-sided rectangles of perimeter 24 cm, find the rectangle with the largest area and the rectangle with the smallest area.

Discussion. This activity is designed to approach conceptually the fundamental geometric characteristic of area of a rectangular shape. The first question to be addressed is: How can one measure area of a rectangle? A not uncommon answer is when a teacher candidate provides the formula: area is the product of length and width. However, defining a concept through a formula can hardly be accepted as a conceptual approach to area of a geometric shape. To define area, one has to define a unit of measurement for area. Such a unit was used in section 10.1 in the context of geoboard activities. This unit is called the unit square –a square of area 1 square unit. If we define a rectangle as a quadrilateral with the pairs of adjacent sides forming right angles, then an integer-sided rectangle with the side lengths n and m can be represented as an $n \times m$ array of unit squares, say, n unit squares going horizontally from left to right and m unit squares going vertically from top to bottom. But here, while one sees the product $n \times m$ which defines area, the product stems from counting unit squares and not from multiplying the side lengths. Likewise, the volume of a right rectangular prism with integer dimensions is understood as the number of unit cubes that such a prism includes. This is similar to measuring perimeter by using a ruler which shows a linear unit of measurement (e.g. 1 inch or 1 cm). So, both perimeter and area are computed by counting the corresponding units of measurement. This position is consistent with the Conference Board of the Mathematical Sciences (2012) ideas about mathematics teachers' "understanding what area and volume are and giving rationales for area and volume formulas" (p. 29).

Having perimeter 24 cm, one has half of the perimeter representing the sum of the side lengths of two adjacent sides. Dividing 24 by 2 yields 12; that is, the sum of two adjacent sides in the family of integer-sided rectangles of perimeter 24 cm is 12 cm. In order to find all rectangles that belong to the given family, one has to decompose the number 12 into the sums of two integers, a task already known to students in Grade 1. In all, there are six decompositions: $12 = 11 + 1, 12 = 10 + 2, 12 = 9 + 3, 12 = 8 + 4, 12 = 7 + 5, 12 = 6 + 6$. That is, the pairs of adjacent sides are: (11, 1), (10, 2), (9, 3), (8, 4), (7, 5), and (6, 6). The corresponding rectangles are shown in Figure 10.12. By counting unit squares included in each rectangle, the values of areas (expressed in cm^2) are displayed to the left of a rectangle: 11 cm^2, 20 cm^2, 27 cm^2, 32 cm^2, 35 cm^2, and 36 cm^2. One can see that a rectangle with congruent adjacent sides (i.e., square) has the largest area. One can try several other even number values for perimeters (in order to have the sum of two adjacent sides an integer) to confirm that square has the largest area. Using informal deduction (Level 2 of the van Hiele model of geometric thinking), that is, making a general statement based on a number of examples in favor of generalization, one can conclude that, given a perimeter of the family of integer-sided rectangles, square always has the largest area and the "skinny" rectangle (that is, the rectangle with the unit side length) always has the smallest area.

But what is we allow for a side length to be a fraction; in particular, to be smaller than the number 1? In that case, one has to define area differently than through counting unit squares (if one wants to find area exactly, not to the accuracy of whole numbers). It is at this point that counting unit squares in a rectangular array through multiplication may be interpreted as multiplying length by width, because the number of unit squares going from left to right is the length and the number of unit squares going from top to bottom is the width (or vice versa). In the case of the pairs introduced above for perimeter 24 cm we have:

$$(11,1)\rightarrow A=11\times 1=11\,cm^2;(10,2)\rightarrow A=10\times 2=20\,cm^2;(9,3)\rightarrow A=9\times 3=27\,cm^2;$$
$$(8,4)\rightarrow A=8\times 4=32\,cm^2;(7,5)\rightarrow A=7\times 5=35\,cm^2;(6,6)\rightarrow A=6\times 6=36\,cm^2.$$

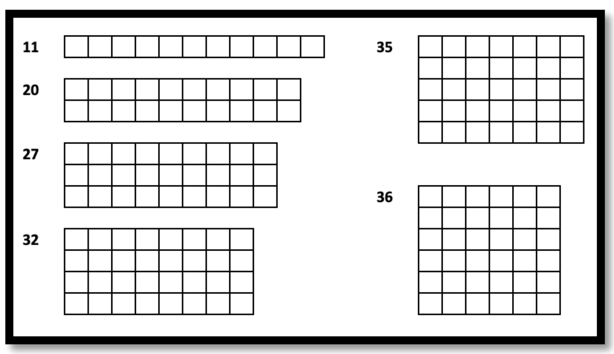

Figure 10.12. The range of areas of integer-sides rectangles with perimeter 24 cm.

In general, for a rectangle with the side lengths *a* and *b*, the area $A=a\times b$. Whereas the last formula is nor rigorously proved, one can at least understand "where a mathematical rule comes from" (Common Core State Standards, 2010, p. 4). Using this rule (i.e., formula), one can find area of a rectangle with perimeter 24 cm and the smaller side length equal to 1/2 cm. We have

$12 = \frac{1}{2} + 11\frac{1}{2}$ and, therefore, $A = \frac{1}{2} \times 11\frac{1}{2} = \frac{1}{2} \times \frac{23}{2} = \frac{23}{4} = 5\frac{3}{4}$. One can see that $5\frac{3}{4} < 11$, that is, by decreasing the smaller side length one decreases area.

Likewise, $12 = \frac{1}{4} + 11\frac{3}{4}$ and, therefore, $A = \frac{1}{4} \times 11\frac{3}{4} = \frac{1}{4} \times \frac{47}{4} = \frac{47}{16} = 2\frac{15}{16} < 5\frac{3}{4} < 11$. By making the smaller side length as small as one wishes, the area of rectangle can also be made as small as one wishes; in other words, no rectangle with the smallest area can be found. For example, the decomposition $12 = \frac{1}{1000} + 11\frac{999}{1000}$ implies $A = \frac{1}{1000} \times 11\frac{999}{1000} \approx 0.01$ and such decrease in the value of area continues as long as one wishes. At the same time, regardless of the side lengths of rectangles with the fixed perimeter, square always has the largest area. This fact can be proved using mathematical tools studied in high school.

Activity 2. Among all integer-sided rectangles of area 36 cm^2, find the rectangle with the smallest perimeter and the rectangle with the largest perimeter.

Discussion. Now, by seeing area of a rectangle as a rectangular array of unit squares, one can find different rectangles with the given area through factoring the number 36 into the product of two integers. We have: $36 = 36 \times 1, 36 = 18 \times 2, 36 = 12 \times 3, 36 = 9 \times 4, 36 = 6 \times 6$. That is, the pairs (36, 1), (18, 2), (12, 3), (9, 4), and (6, 6) represent adjacent side lengths of the family of integer-sided rectangles of area 36 cm^2. All the rectangles with perimeters equal to 74 cm, 40 cm, 30 cm, 26 cm, and 24 cm, respectively, are shown in Figure 10.13. For example, 74 = 36 + 1 + 36 + 1. Likewise, other perimeters can be calculated. Note that any integer, 36 included, can be factored in two factors one of which is a not an integer in infinite number of ways, like $36 = \frac{1}{10} \times 360, 36 = \frac{1}{100} \times 3600$, and so on. One can see that the rectangle with the side lengths represented by the pair (6, 6) has the smallest perimeter and the rectangle with the side lengths represented by the pair (36, 1) has the largest perimeter. That is, the rectangle with congruent adjacent sides (i.e., square) has the smallest perimeter, 24 cm; and the rectangle with the smallest possible integer side (a "skinny" rectangle) has the largest perimeter, 74 cm. However, already the factorization $36=\frac{1}{10}\times 360$ corresponds to the rectangle with side lengths 1/10 cm and 360 cm, the perimeter of which is equal to $\frac{1}{10}+360+\frac{1}{10}+360=720\frac{1}{5}>74$. Likewise, the factorization $36=\frac{1}{100}\times 3600$ brings about the rectangle with the side lengths 1/100 and 3600, the perimeter of which is equal to $\frac{1}{100}+3600+\frac{1}{100}+3600=7200\frac{1}{50}>720\frac{1}{5}>74$.

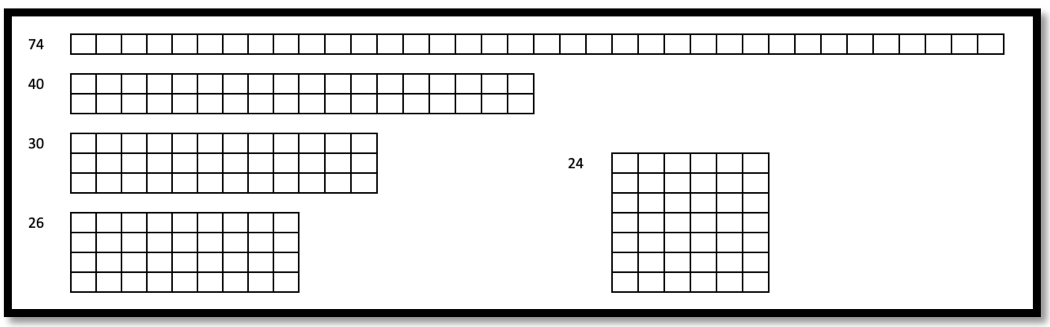

Figure 10.13. The range of perimeters of integer-sides rectangles of area 36 cm^2.

That is, by allowing for a smaller side length of a rectangle of area 36 cm^2 to be as small as one wishes, perimeter of rectangle can be made as large as one wishes. In other words, given area, no rectangle with the largest perimeter can be found. At the same time, by trying other (perfect square) values of area and using informal deduction (Level 2 of the van Hiele model of geometric thinking), one can conclude that given area, square always has the smallest perimeter. This geometric fact can be nicely applied to real life. Namely, if one buys a rectangular piece of land with a given area and wants to minimize expenses in building a fence around the land, one has to buy a square piece of land. Likewise, when having a certain amount of fence, the largest rectangular piece of land to be fenced with this amount is the square.

CHAPTER 11: ELEMENTS OF PROBABILITY THEORY AND DATA MANAGEMENT

> "Well prepared beginning teachers of mathematics at the upper elementary level effectively use technology tools, physical models, and mathematical representations to build student understanding of the topics at these grade levels" (Association of Mathematics Teacher Educators, 2017, p. 74).

11.1 Introduction

Teaching basic ideas of probability theory begins at the early elementary level by asking questions about chances of something to happen. What are the chances of rain on the Labor Day? What are the chances of snow on the last day of school year? It is important for a child to understand that in the above two cases the chances are quite different. In a non-weather context, what are the chances to pick up (without looking) a penny from a box with three pennies and one dime? Here, it is important for a child to understand that without looking an outcome is uncertain; otherwise, an outcome is certain. Still, within this uncertainty of experimental situation, there should be more chances for a penny (just as there are more chances for rain in September than for snow in June). As a result, a child should appreciate experimental character of situations associated with uncertain outcomes.

Teaching about chances can be supported by the appropriate use of physical and digital teaching tools. At the upper elementary level, students "use proportionality and a basic understanding of probability to make and test conjectures about the results of experiments and simulations" (National Council of Teachers of Mathematics, 2000, p. 248) and "learn about the importance of representative samples for drawing inferences" (Common Core State Standards, 2010, p. 46).

In the study of probability, experiments with equally likely outcomes are of a special importance. For example, tossing a fair coin and rolling an unbiased die can be set as experiments with equally likely outcomes. It is equally likely to have either head or tail when tossing a coin and either one or six when rolling a die. An Italian mathematician Gerolamo Cardano (1501–1576) is credited with the definition of the concept of probability as the ratio (see Chapter 9) of the number of favorable outcomes to the total number of *equally likely* outcomes within a certain experimental situation. This classic definition is commonly accepted nowadays in school mathematics. While saying that there are three chances out of four to pick up a penny in the above situation can be seen as an intuitive conclusion, comparing chances of two situations requires numerical measurement of chances. When solving probability problems with equally likely random outcomes, this definition gives an applied flavor to fractions in the range [0, 1] as numerical characteristics of what is considered probable and makes it possible to compare chances by using proper fractions (or their decimal equivalents). Thus, according to Cardano, the chances (alternatively, likelihood or probability) to pick up a penny is the fraction 3/4. But chances (probabilities) related to different situations, in order to be compared, have to be computed (measured) first. Mathematical actions of that kind should not be considered through the simplistic lenses, like measuring perimeters and areas of basic geometric shapes through selecting appropriate measurement tool of the modern-day geometry software. For example, whereas chances to select a penny from the boxes with three pennies and one dime, and five pennies and two dimes, respectively, are close, the chances for the former box are higher as 3/4 > 5/7. And until one knows how to express chances through a common fraction and how to compare fractions, the comparison of chances can only be done intuitively or through an experiment.

When tossing a coin and knowing with certainty that it would not fly, one, however, cannot predict how exactly it would fall. But through tossing a coin many times and recording the results, one can recognize how randomness turns into regularity. Indeed, one can check to see that out of 100 tosses, a coin would turn up head or tail somewhere in the range [40, 60]; that is, an *experimental* probability of head or tail, computed as the ratio of the number of times a desired outcome occurred to the total number of trials, would be a number in the range [0.4, 0.6].

	A	B	C	D	E	F	G	H
1							596	595
2						0.0001	0.0596	0.0595
3	1	1	1	1		HHHH		
4	1	0	0	1				
5	0	1	1	1				
6	0	0	0	1				
7	0	1	0	1				
8	0	1	1	0				
9	0	1	1	1				
10	1	1	0	0				
11	1	1	0	0				
12	0	0	1	1				
9998	1	1	1	1		HHHH		
9999	0	1	1	0				
10000	1	0	1	0		HTHT		
10001	0	1	0	1				
10002	1	0	0	0				

Figure 11.1. HHHH vs. HTHT.

A more sophisticated experimentalist can attempt to measure the results of possible outcomes when a coin is tossed, say, four times, in a large series of experiments. A surprising result, which first can be established experimentally by using a spreadsheet, is that having four heads (HHHH) has the same probability as having first head, then tail, and then again head and tail (HTHT). The spreadsheet of Figure 11.1 shows that as a result of 10,000 (simulated) tosses of a coin the incident HHHH happened about as many times as the incident HTHT (616 vs. 621); in other words, the chances for HHHH and for HTHT are measured, respectively, by two fractions

(616/10000 = 0.0616 and 621/10000 = 0.0621) differing by 5/10000 = 0.0005 (Figure 11.1, cell F2). Multiple repetitions of this computational experiment would demonstrate that both incidents have almost equal chances within a large series of experiments.

12.2 Randomness and sample space

In this section, several basic concepts associated with the probability strand will be explained. The first one is randomness – a characterization of the result of an experiment that is not possible to predict but, nonetheless, is often possible to measure on the scale from zero to one, with zero assigned to something impossible (e.g., a coin flies when tossed in the still air) and one assigned to something certain (e.g., a coin falls when tossed in the still air). One can say that randomness does not lead to a credible pattern, although it is often difficult to conclude whether there is no pattern in a sequence of events. Nonetheless, thinking about randomness in a very informal way, can help one understand the "difference [between] predicting individual events and predicting patterns of events" (Conference Board of the Mathematical Sciences, 2001, p. 23). In the case of tossing a coin, whereas one cannot predict how exactly it would fall, it is not impossible to have head and tail alternating in, say, five tosses, or even to have five heads or tails in a row. Yet, due to experience, one can predict a pattern in the sequence of 100 tosses. Whatever a prediction, the question to be answered is how to measure the likelihood (chances) of the prediction.

The need for measuring chances leads to the concept of the sample space of an experiment with a random outcome that is defined as the set of all possible incidents associated with this experiment. A simple example is the sample space of rolling a six-sided die (with the number of spots on the sides ranging from one to six) comprised of six outcomes: $\{1, 2, 3, 4, 5, 6\}$. Assuming that we deal with a fair (unbiased) die, all outcomes may be considered equally likely. Under this assumption, one can say that there is one chance out of six to cast any of the six numbers (spots).

N\D	0	1	2
0	25	15	5
1	20	10	0
2	15	5	
3	10	0	
4	5		
5	0		

Fig. 11.2. A sample space of changing a quarter into pennies, nickels and dimes.

A more complicated example is the sample space of an experiment of changing a quarter into pennies, nickels and dimes (see the last task mentioned in Chapter 6). The chart of Figure 11.2 shows a twelve-element sample space, where numbers in the top row and the far-left column show the ranges for dimes and nickels, respectively, and the remaining numbers show the corresponding number of pennies in a change. For example, the triple (5, 2, 1) stands for five pennies, two nickels, and one dime, so that $5 \cdot 1 + 2 \cdot 5 + 1 \cdot 10 = 25$. Alternatively, the sample space can be described as the following set of twelve triples of numbers $\{$(25, 0, 0), (20, 1, 0), (15, 2, 0), (10, 3, 0), (5, 4, 0), (0, 5, 0), (15, 0, 1), (10, 1, 1), (5, 2, 1), (0, 3, 1), (5, 0, 2), (0, 1, 2)$\}$, where each triple describes the number of pennies, nickels and dimes, respectively, in a change. Note that there is no reason to assume that it is equally likely to get any combination of the coins from a change-making device and Figure 11.2 represents a sample space with not equally likely outcomes. Therefore, given a change-making device, it is only experimentally that one can determine chances for a specific combination of coins in a change. Alternatively, for the purpose of problem solving, one can assume that it is equally likely to get any combination of the coins in a change out of the total twelve. Under this assumption, one can say that the probability of not having pennies in the change is equal to 1/4 (as 3/12 = 1/4).

11.3 Different representations of a sample space

A sample space of an experiment with random outcomes may have different representations. Here is an example: the sample space of tossing two coins can be represented in the form of a table and a tree diagram as shown in Figure 11.3 and Figure 11.4, respectively. Note that the outcomes of the two tosses are independent; that is, the outcome of the second toss does not depend on what happened on the first toss. This independence is reflected in the very form of the tree diagram: each of the two possible outcomes of the first toss affords the same two outcomes for the second toss. This, however, is not always the case and in more complex situations, drawing a tree diagram to represent a sample space may be a difficult proposition.

An outcome of an experiment is an element of its sample space. For example, the sample space of the experiment of tossing two coins shown in Figure 11.3 (or Figure 11.4) consists of four outcomes. Further, outcomes may be combined to form an event. For example, the outcomes HH and TT form an event that both coins turn up the same. Depending on specific conditions, the outcomes of an experiment may or may not be equally likely. All the outcomes of the experiments of tossing coins or rolling dice described in the above examples are considered equally likely, assuming that one deals with fair coins of unbiased dice. An assumption about equally likely outcomes is a theoretical assumption – after all, when we deal with a coin (or a die), we *assume* that it is a fair coin (or an unbiased die).

HH	TH
HT	TT

Figure 11.3. The sample space of tossing two coins in the form of a table.

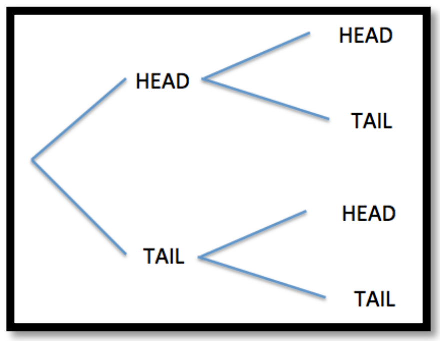

Figure 11.4. The sample space of tossing two coins in the form of a tree diagram.

Outcomes are not always equally likely. Indeed, when having a holder with three red and two black markers, it is not equally likely (without looking) to pick up either a red or a black marker. Likewise, events may be independent and dependent. In the case when from the first draw of a marker from the holder, the marker is returned to the holder, the outcome of picking up a red marker on the second draw does not depend on which color marker was picked up on the first draw. At the same time, if the marker is not returned, the outcome of picking up a red marker on the second draw depends on what happened on the first draw.

The sample space of rolling two dice and recording the total number of spots on two faces can be represented in the form of an addition table (Figure 11.5) from where it follows that the highest chances are for having seven as the total number of spots on two dice. Here, one can connect probability concepts to the concept of integer partitions into summands (see Chapter 4). Indeed, whereas twelve has more partitions than seven, in the context of rolling two dice partitions are limited to the summands that are not greater than six. Under such condition, the closer a number to six is, the more ordered partitions into two integer summands exist. For example, the sum 12 has only one possibility, 12 = 6 + 6, and the sum 7 has six possibilities, 7 = 1 + 6 = 6 + 1 = 2 + 5 = 5 + 2 = 3 + 4 = 4 + 3.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure 11.5. The sample space of rolling two dice.

11.4 Understanding chances in a computer environment

The following two problems aim at elementary school students' learning to compare chances before they are taught fractions which serve as the tools for computing and comparing chances. That is, the students learn to compare chances other than through intuition of an experiment.

Problem 1. Billy wants to eat red M&Ms only. There are two bags of M&Ms available. If there are 4 red plain out of 10 total in the first bag, and 10 red peanut out of 20 total in the second bag, for which bag does Billy have more chances to get a red M&M? What are these chances? Use sliders to create the bags of M&Ms. You may use the bar graphs to answer the first question.

Problem 2. Billy ate (by using sliders) one red M&M from each bag. How many more red peanut M&Ms does Billy have to eat in order to give Mary more chances to get a red plain M&M than a red peanut M&M? Use sliders to create new bags of M&Ms. What are Mary's chances for getting a red M&M for both bags?

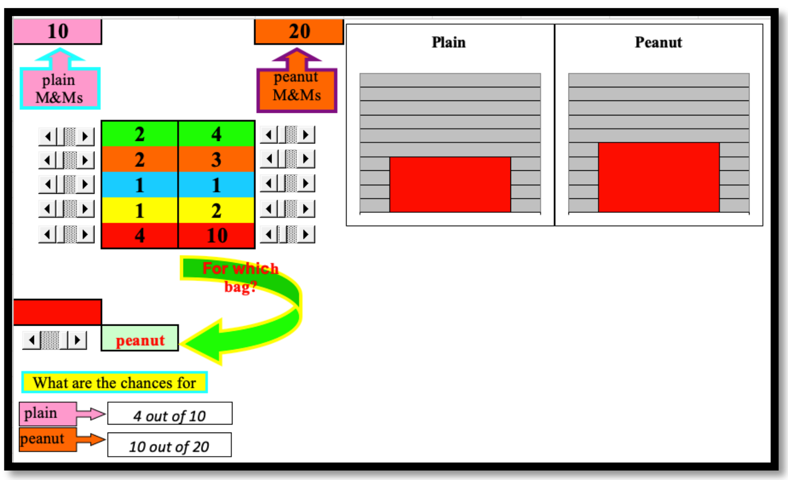

Figure 11.6. Chances for peanut M&M are higher.

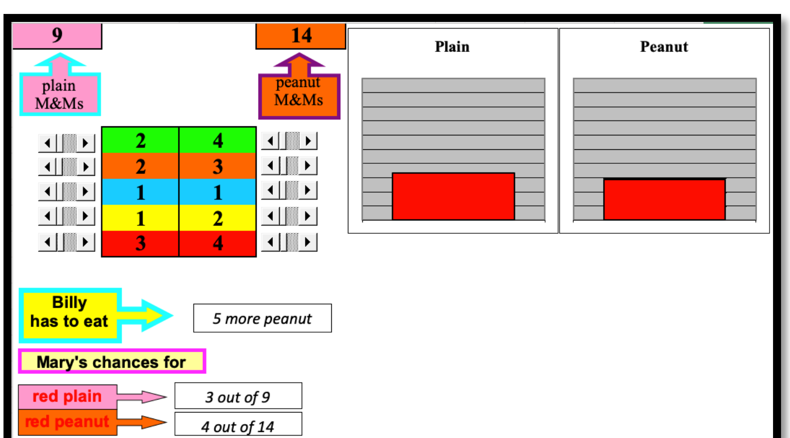

Figure 11.7. Making chances for plain M&M higher.

From the quantitative perspective, Problem 1 requires the comparison of two fractions, 4/10 and 10/20. When students do not know fractions and, moreover, do not know how to compare fractions, they have no idea for which bag the chances to pick up a red M&M are higher. An intuitive grasp of fractions as quantities can be developed using their iconic representations. To this end, spreadsheet-based bar graphs can be employed for an intuitive, visual comparison of fractions. Such graphs are the analogues of those used to represent and compare whole numbers (see Chapter 1). In such a way, students can acquire an informal, intuitive understanding of quantitative relationships between chances before they actually learn about formal ways of expressing and comparing chances using fractions. In other words, the comparison of chances can be developed first by using mathematical visualization in the iconic environment of a spreadsheet. The use of fractions as symbolic representations of the chances develops later on the foundation provided by visualization. That is, students can build their leaning of formal operations with fractions on the knowledge they received in a fun and informal, yet mathematically meaningful, way. Figures 11.6 and 11.7 present students' (correct) responses to questions posed in Problem 1 and Problem 2, respectively. For more details on this project see Abramovich, Stanton, and Baer (2002).

11.5 Fractions as tools in measuring chances

Application of arithmetic to geometry brings about the idea of using a numeric measure of the chances (probability) of an event. One of the ways to introduce the concept of fraction as an applied tool is through using fractions as a measure of the chances (or probability) of an event.

The idea of a geometric representation of a fraction enables its use as a means of finding the probability of an event (alternatively, measuring its chances).

Consider equally likely outcomes of the experiments with tossing a coin and rolling a die. The meaning of the words *equally likely* can be given a geometric interpretation through representing such outcomes as equal parts of the same whole. In the case of coins, we have four equally likely outcomes, which can be represented in the form of a rectangle divided into four equal parts (Figure 11.3) so that each part is represented by a unit fraction 1/4. It is this fraction that can be considered as the value of the likelihood (probability) of any of the four (equally likely) outcomes. Dividing the rectangle in four equal parts as shown in Figure 11.3 demonstrates two things: (i) a table representation of the sample space of tossing two coins; (ii) according to the area model for fractions (Chapter 7), each of the four parts represents the product $(1/2)(1/2)$, where each factor is the probability of head/tail for each of the two tosses. Extending the number of tosses beyond two, one can measure the probability of alternating heads and tails in, say, five tosses by the number $(\frac{1}{2})^5 = \underbrace{\frac{1}{2}\cdot\frac{1}{2}\cdot\ldots\cdot\frac{1}{2}}_{5\ times}$. One can see that the same number also measures the probability of having five heads (or tails) in a row.

In the case of rolling two dice, each of the 36 outcomes that comprise the table-type sample space (Figure 11.5) is equally likely and, therefore, the fraction 1/36 is the measure of each outcome. At the same time, the likelihood of the event that after rolling two dice, the result is either eight or nine spots on both faces is measured by the fraction of the table filled with either eight or nine. As the two numbers appear nine times in the (addition) table, the probability of this event is 9/36 or 1/4. In that way, one can conclude that the chances of having HH after tossing two coins are the same as having the sum of either eight or nine on both faces when rolling two dice.

Apparently, without using fractions as measuring tools for chances (probabilities) this conclusion would not be possible.

11.13 Graphic representations of numeric data

Suppose a die was rolled 20 times and the number of spots on a die assumed the following twenty values 1, 6, 4, 1, 5, 6, 1, 2, 3, 3, 5, 6, 6, 2, 3, 5, 6, 4, 4, 6. These values can be written in the non-decreasing order 1, 1, 1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6, 6, and then organized in different graphical formats. One such format is a graph called a line plot (Figure 11.8). Reading the graph from left to right, one can see that the first three dots indicate the number of occurrences of 1's, the next two dots – the 2's, …, and, finally, the last six dots – the number of occurrences of 6's. Another graphic representation is called a histogram (Figure 11.9). Reading the histogram, one can recognize that the height of each of the six bars indicate the number of occurrences of each of the six spots on the face of a die. As mentioned by the Association of Mathematics Teacher Educators (2017), "Well-prepared beginners of mathematics in Pre-K to Grade 2 understand that the foundations of statistical reasoning begin with collecting and organizing data" (p. 52).

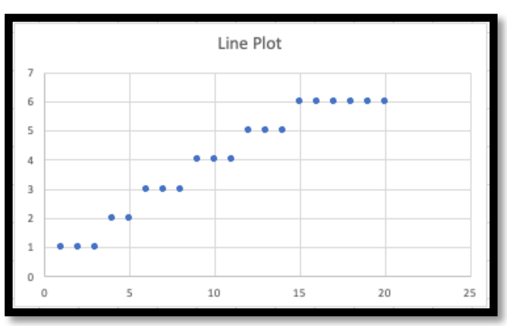

Figure 11.8. A line plot constructed with *Excel*.

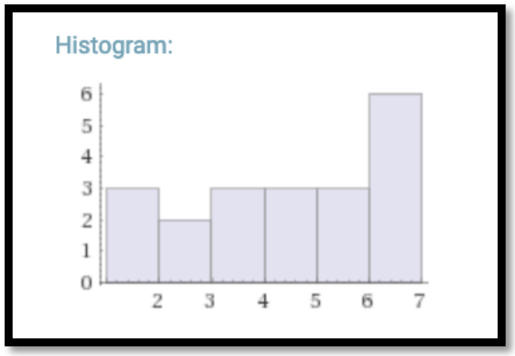

Figure 11.9. A histogram constructed with *Wolfram Alpha*.

0	9	5			
1	0				
2	4	2			
3	5	7			
4	4	5			
5	8	7	3		
6	0				
8	1	1	4	9	9
9	4	5			

Figure 11.10. Stem and leaf plot.

Numeric data can also be presented in a stem and leaf plot format. For the set of 20 one-digit numbers the stem may be defined as one of the numbers and the leaf may be defined as the number of occurrences of that numbers in the set. However, more common is to have one number representing the stem and having several leaves associated with this stem, something that a set of one-digit numbers does not follow. For example, in plotting the set of twenty integers 58, 35, 60, 57, 44, 53, 94, 81, 81, 24, 10, 95, 9, 45, 37, 5, 84, 89, 22, 89, the stem of the plot may be defined as the tens' digits and leaves of the plot are the ones' digits. The corresponding stem and leaf plot is shown in Figure 11.10. In the presence of three-digit numbers, a stem may be the face value for the largest place value and the leaves are two-digit numbers. For example, the line 3| 0 7 45 89 may be read as the set {3, 37, 345, 389}.

11.14 Measures of central tendency

Suppose that five children decided to contribute with as much money as each of them had towards a common goal of buying a new basketball. Their individual contributions were \$3, \$3, \$5, \$8, and \$11 totaling \$30. Also, they wanted to know how could \$30 be collected from equal contributions and divided the number 30 by 5 to get 6. Put another way, $6 = \frac{3+3+5+8+11}{5}$. Statistically speaking, the number 6 can be used as a characteristic of their contributions; it is called the mean (or average) of the numbers 3, 3, 5, 8, and 11. Note that none of the five numbers is equal to six. Yet, the number 6 can be used to describe the five numbers in terms of fair sharing of moneys. In general, given the set of *n* numbers $x_1, x_2, ..., x_n$, the mean value, $\bar{x}$, is defined as

$$\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n}.$$

One can also look at the above five contributions to see whether some children from the group contributed the same amount into the purchase of a basketball. An observation can be made that two children each contributed \$3, something that may be considered the most common contribution among the five. In statistics, this number is called the mode.

In general, if in the set of numbers there exist a single number or several numbers that occur most frequently, each one is called the mode. That is, the set of numbers may have no mode or may have one or several modes. Often, numbers collected for statistical evaluation have to be arranged in the increasing (or non-decreasing) order. In such case, depending on the total of numbers collected, there is either a number in the middle of the list or there are two neighboring numbers equidistant from the first and the last numbers on the list. Either the number in the middle or the average of the two numbers in the middle of the list is called the median. For example, for the list 3, 3, 5, 8, 11 the median is 5 and for the list 2, 3, 4, 6, 7, 8 the median is 5 as well. That is, like the mean, the median may or may not be a part of the list. In statistics, the mean, median, and mode are considered the most basic characteristics through which a data set can be described, and they are called the measures of central tendency. These measures might be good characteristics to be used in one case and weak characteristics to be used in another case.

Indeed, whereas the set of six numbers {2, 3, 4, 6, 7, 8} does not include their mean and median (which are the same), the number 5 does not deviate much from either the smallest or the greatest. At the same time, the set of six numbers {0, 1, 2, 2, 2, 23} has the same mean, 5, as the former set; yet the number 5 deviates significantly from its largest number. This observation implies that measures of central tendency may not be sufficiently good characteristics of a set of numbers in terms of their (the numbers') deviation from the three characteristics. Therefore, new tools of statistical analysis have to be developed.

APPENDIX

Activity sets

ACTIVITY SET 1: CONCEPTUAL SHORTCUTS, THE ORDER OF OPERATIONS, AND ARITHMETIC ON PLACE VALUE CHARTS

1) How can one do addition and subtraction in the expression 53 + 9 – 7 through a conceptual shortcut? Explain your answer.

2) How can one do addition and subtraction in the expression 37 + 9 – 6 through a conceptual shortcut? Explain your answer.

3) Formulate two similar addition and subtraction problems to be solved through a conceptual shortcut. Solve your problems.

4) Multiply 5 by 7 through a conceptual shortcut using commutative property of multiplication. Explain the meaning of the conceptual shortcut used here.

5) Multiply 55 by 11 through a conceptual shortcut using a geometric representation of the product. Draw a picture.

6) Find the sum 1 + 2 + 3 + ... + 10 through a conceptual shortcut. Demonstrate (on picture) how to use manipulatives in support of the conceptual shortcut. Connect this summation to the history of mathematics.

7) Using pictures, solve the following problem: If 2 twiggles and 5 glogs have 18 legs, how many legs does each creature have? Alternatively, if one spent $18 on buying notebooks and pens each of which costs $2 and $5, respectively, how many notebooks and how many pens did one buy? Now, solve the equation $2x + 5y = 18$ in whole numbers using a conceptual shortcut. Here the variables x and y stand for the number of legs each creature has; alternatively, x and y

are the prices of a notebook and a pen. What if we replace 18 by 24? Does the corresponding equation (with 24 in the right-hand side) have only one solution? Why or why not?

8) Solve the following problem through a conceptual shortcut: If, preparing for a party, one spent \$150 on water sets and pizzas each of which costs \$10 and \$13, respectively, how many water sets and how many pizzas did one buy?

9) Formulate a similar problem that leads to an equation that can be solved through a conceptual shortcut.

10) Find the values of the following numeric expressions using a calculator, a spreadsheet and *Wolfram Alpha*:

$32 \div 8 + 24; 32 \div (8 + 24); 24 \cdot 2 \div 2; 24 \cdot (2 \div 2); 64 \cdot 16 \div 4; 64 \div 16 \cdot (10 + 6); 64 \div 16 \cdot (6 - 2).$

11) Find 23 + 16, 27 + 35 using a place value chart.

12) Find 26 – 13, 35 – 17 using a place value chart.

13) A third grade student struggles with rounding an integer to the nearest ten and hundred places. Use manipulatives to help the student with the idea of rounding to the nearest benchmark integer (10 and 100 included).

14) Put the above activities in the context of Common Core State Standards for Mathematics (http://www.corestandards.org/Math/).

ACTIVITY SET 2: CONNECTING MATHEMATICAL PRACTICE WITH MATHEMATICAL CONTENT OF ADDITIVE DECOMPOSITIONS OF INTEGERS

.

Task 1. If the key 3 on your calculator were broken, how could you find the sum 354 + 531 + 453?

1.1) Is there more than one way of doing that? Show how this problem can be presented on a place value chart using multicolored counters.

1.2)Assuming that you do need a calculator for adding two two-digit numbers and the keys 3, 4 and 8 were broken, how could you find the sum 13 + 31 using two natural number addends? How many different representations of this sum through two natural numbers are there? Write down all such representations. Do the same for the sum 12 + 21 with the key 2 broken.

1.3) How many ways can one put 8 cookies on two plates without regard to order of the plates, no plate having three cookies, and with at least one cookie on each plate?

1.4) How many ways can one put 8 cookies on 3 plates without regard to order of the plates, no plate having three cookies, and with at least one cookie on each plate?

1.5) How many ways can one put 10 cookies on 4 plates without regard to order of the plates, no plate having three cookies, and with at least one cookie on each plate?

Task 2. Find all ways to add consecutive counting numbers in order to reach sums in the range 1 through 16. (For example, 3, 4, and 5 are consecutive numbers and 3+4+5=12. There are also sums comprised of two, four, and five addends).

2.1) Use a spreadsheet to make a table displaying all the sums.

2.2) Which integers (sums) appear in the table more than one time?

2.3) Which integers from the range [1, 16] do not appear in this table and what is special about them?

2.4) What can be said about the sums of two consecutive counting numbers?

2.5) What can be said about the sums of three consecutive counting numbers?

2.6) What can be said about the sums of four consecutive counting numbers?

2.7) Use manipulatives (e.g., square tiles) to support your answers to questions 4) – 6).

2.8) Draw representations for the numbers 9 and 15 through the sums of consecutive counting numbers using counters.

Task 3. Put the activities of Tasks 1 and 2 in the context of Common Core State Standards for Mathematics (http://www.corestandards.org/Math/)

and/or New York State Next Generation Mathematics Learning Standards

(http://www.nysed.gov/common/nysed/files/programs/curriculum-instruction/nys-next-generation-mathematics-p-12-standards.pdf),

and/or Standards for Preparing Teachers of Mathematics by Association of Mathematics Teacher Educators (https://amte.net/sptm).

ACTIVITY SET 3: ACTIVITIES WITH MULTIPLICATION AND ADDITION TABLES

.

1) Use *Wolfram Alpha* to construct a 5x5 multiplication table (type in the input box of the program “5x5 multiplication table” or just “5 multiplication table”).

2) Show three ways to find the sum of all numbers in the 5x5 multiplication table.

3) Using any of the three ways, find the sums of numbers in 4x4, 3x3, 2x2, and 1x1 multiplication tables.

4) Make an organized list of the (five) sums and enter it in the input box of *Wolfram Alpha* in order to find the continuation of this sequence. What is the first number generated by *Wolfram Alpha* in response?

5) Use one of the ways of finding the sum in item 2) in order to find the sum of numbers in 6x6 multiplication table and check if your finding corresponds to the number generated by *Wolfram Alpha*.

6) Do activities 1) – 5) for the corresponding addition tables.

7) Put these activities in the context of Common Core State Standards for Mathematics (http://www.corestandards.org/Math/)

and/or New York State Next Generation Mathematics Learning Standards (http://www.nysed.gov/common/nysed/files/programs/curriculum-instruction/nys-next-generation-mathematics-p-12-standards.pdf),

and/or Standards for Preparing Teachers of Mathematics by Association of Mathematics Teacher Educators (https://amte.net/sptm).

ACTIVITY SET 4: USING TECHNOLOGY IN POSING PROBLEMS

Task 1. It takes 46 cents in postage to mail a letter. A post office has stamps of denomination 5 cents, 7 cents, and 10 cents. How many combinations of the stamps could Ada buy to send a letter if the order in which the stamps are arranged on an envelope does not matter? What are those combinations? How can you prove that you found all combinations of the stamps?

Task 2. How many ways can one make a quarter out of pennies, nickels, and dimes?

Task 3. Use the spreadsheet labeled "Problem Generator" to generate and formulate a problem similar to Problem 1 and Problem 2. List all solutions to your own problem. Show a picture of your spreadsheet. Develop and explain a systematic way of solving your problem when technology (the spreadsheet) is not available. Note: a teacher uses technology to pose a problem for students to solve it without technology. Technology saves teacher's time in selecting different numeric data that best suits their specific classroom.

Read Abramovich, S., and Cho, E. (2008). On mathematical problem posing by elementary teachers: The case of spreadsheets. *Spreadsheets in Education*, 3(1), pp. 1-19, Available on-line at https://sie.scholasticahq.com/issue/791 and answer the following questions related to the problem you posed.

1. How many solutions (different answers) does your problem have?

2. Do you think that the spreadsheet generated all solutions to your problem? Why or why not?

3. What grade level(s) is your problem appropriate for?

4. Do you expect young children to find all solutions to your problem? Explain your answer.

5. Is your problem numerically coherent? Why or why not?

6. Is your problem pedagogically coherent? Why or why not?

7. Is your problem contextually coherent? Why of why not?

8. What do you think about the role that computing technology can play in problem posing by elementary teachers?

ACTIVITY SET 5: LEARNING TO MOVE FROM VISUAL TO SYMBOLIC

1) Using diagrams (see section 2.3 of the Course Materials) show that $5\times1+5\times10=5\times2+5\times9=5\times3+5\times8$. What is special about the pairs (1, 10), (2, 9) and (3, 8).

2) On the diagram of Figure 1, do the following:

2.1) Count the total number of circles;

2.2) Count (from top to bottom) the number of circles in the first four rows;

2.3) Count (from top to bottom) the number of circles in the first three rows;

2.4) Find the difference between the numbers found in items 2) and 3);

2.5) Count the number of shaded circles.

2.6) What is special about the numbers found in items 1), 2), 3) and 5)?

2.7) Show on the diagram of Figure 1 that $15=3\times(10-6)+3$.

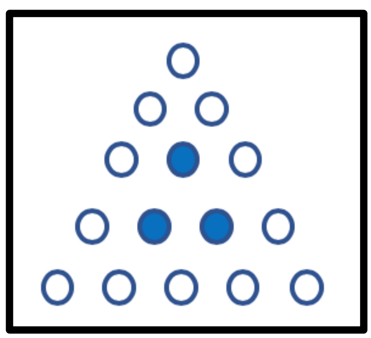

Figure 1. Circles forming a triangular shape.

3) Draw a 5 by 5 square and show that 25 = 1 + 3 + 5 + 7 + 9.

4) Using a square formed by small circles (similar to Figure 1) show that $25=3\times(16-9)+4$.

5) Put these activities in the context of Common Core State Standards for Mathematics (http://www.corestandards.org/Math/) .

ACTIVITY SET 6: COMPARISON OF FRACTIONS ON A PRE-OPERATIONAL (CONCEPTUAL) LEVEL

1) Compare fractions (which one is bigger) using two-dimensional model: 1/2 or 1/3, 2/5 or 3/7, 3/4 or 4/5? Draw pictures demonstrating comparison.

2) How can one compare 3/4 and 4/5 using fraction circles without recourse to the two-dimensional model? Draw pictures using the program *Fraction Circles* in making this demonstration. What is special about the fractions 3/4 and 4/5? Give three examples of similar pairs of fractions.

3) Divide 5 into 3 using divided-divisor context and pose a problem that leads to this division.

4) Divide 5 into 3 using part-whole context and pose a problem that leads to this division.

5) A benchmark fraction is a unit fraction. Place the fraction 3/16 between two consecutive benchmark (unit) fractions. Show your work.

6) Place the fraction 31/61 between two consecutive benchmark fractions. Show your work.

7) Convert 3/16 into an Egyptian fraction. Convert 31/61 into an Egyptian fraction. You may use *Wolfram Alpha*, but you have to show your work (informed by *Wolfram Alpha*).

8) Put these activities in the context of Common Core State Standards for Mathematics (http://www.corestandards.org/Math/)

and/or New York State Next Generation Mathematics Learning Standards (http://www.nysed.gov/common/nysed/files/programs/curriculum-instruction/nys-next-generation-mathematics-p-12-standards.pdf),

and/or Standards for Preparing Teachers of Mathematics by Association of Mathematics Teacher Educators (https://amte.net/sptm).

ACTIVITY SET 7: FROM PRE-OPERATIONAL TO OPERATIONAL LEVEL OF DEALING WITH FRACTIONS

1) Complete operations step-by-step using pictures and describe what you see on pictures using fractional notation:

1a) 3/8+1/4 (using fraction circles);

1b) 5/6-1/2 (using fraction circles; is there more than one numeric answer?);

1c) 2/3+1/6 (using rectangles);

1d) 1and 2/3 -1/6 (using set model; is there more than one form of numeric answer?).

2) Formulate a word problem that leads to an operation in each case, 1a) through 1d). Use computer program *Fraction circles* for 1a) and 1b) and *Power Point* for 1c) and 1d).

3a) If the strip  is 5/3, draw the strip which is the whole.

3b) If the strip  is the whole, draw the strip which is 7/4.

3c) If the set of counters shown in Figure 1 is 2/5 of a set, how many counters are in the whole set?

3d) If the set of counters shown in Figure 2 is one whole, how many counters are in 2/3 of the set?

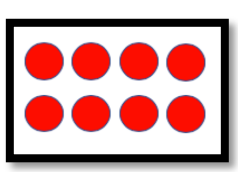

Figure 1.

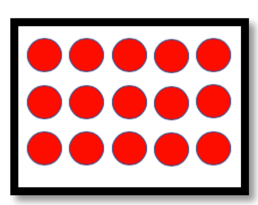

Figure 2.

4) Using area model for fraction, multiply fractions on pictures:

4a) 3x(2/3) and show how formal multiplication stems from this picture;

4b) (2/5)x2 and show how formal multiplication stems from this picture;

4c) (3/8)x(2/5) and show how formal multiplication stems from this picture;

4d) (5/6)x(4/5) and show how formal multiplication stems from this picture.

5) Formulate a word problem that leads to an operation in each case, 4a) through 4d).

6) Put these activities in the context of Common Core State Standards for Mathematics (http://www.corestandards.org/Math/)

and/or New York State Next Generation Mathematics Learning Standards (http://www.nysed.gov/common/nysed/files/programs/curriculum-instruction/nys-next-generation-mathematics-p-12-standards.pdf),

and/or Standards for Preparing Teachers of Mathematics by Association of Mathematics Teacher Educators (https://amte.net/sptm).

ACTIVITY SET 8: PIZZA AS A CONTEXT FOR LEARNING FRACTIONS

1) Divide three identical circular pizzas among five people fairly using dividend-divisor context and Egyptian fraction Greedy algorithm. Compare the number of pizza pieces obtained in each case. Use computer programs *Fraction Circles* and *Wolfram Alpha*.

2) Divide five identical circular pizzas among six people fairly using dividend-divisor context and Egyptian fraction Greedy algorithm. Compare the number of pizza pieces obtained in each case. Use computer programs *Fraction Circles* and *Wolfram Alpha*.

3) A pizza is cut into three different pairs of equal pieces: (1/4, 1/4), (1/6, 1/6) and (1/12, 1/12).

3a) Show this division by using computer program *Fraction Circles*.

3b) These pieces can be used separately to make the full pizza by repeating the first pair twice, the second pair three times, and the third pair six times.

3c) Find other ways to similarly cut pizza into three pairs of equal pieces out of which the whole pizza can be made.

3d) How can one cut pizza into four such pairs?

ACTIVITY SET 9:

WORD PROBLEMS WITH FRACTIONS, PERCENTAGES, AND RATIOS

Problem 1. Anna has six bottles of apple juice. If the serving glass is 3/4 of a bottle, how many servings can she make out of the bottles? Show your solution using a picture. Then provide a formal arithmetical solution.

Problem 2. John has five bottles of orange juice. If the serving glass is 3/4 of a bottle, how many servings can he make out of the bottles? Show your solution using a picture. Then provide a formal arithmetical solution.

Problem 3. Mary has $9\frac{3}{4}$ pounds of flour. It takes $3\frac{1}{4}$ pounds to make a cake. How many cakes can she make? Show your solution using a picture. Then provide a formal arithmetical solution.

Problem 4. A turtle has crept the distance of $3\frac{1}{3}$ km. If it crept $\frac{2}{3}$ km per day, how many days did it take the turtle to creep the whole distance? Solve this problem as measurement and as a missing factor equation by using area model. Show your solution using a picture. Then provide a formal arithmetical solution.

Problem 5. A college has to pave two parking lots, A and B, shaped as squares of sides $85\frac{2}{5}$ meters and $98\frac{2}{5}$ meters, respectively. It is known that 4/5 of lot A and 5/8 of lot B need new pavement. Which parking lot needs more pavement? Use *Wolfram Alpha* for calculations.

Problem 6. Leila paid for her new laptop $1500. It was on sale for 6.25% discount. How much money did she save through this discount?

Problem 7. Jeremy is one of only 6 boys in the class of 16 children. What is the boys to girls ratio in his class? After Christmas break, Jeremy moved with his family to another state. What is the girls to boys ratio in this class after the break?

ACTIVITY SET 10: PROBABILITY AND DATA MANAGEMENT

1. Two dice are rolled. Find the theoretical probability that the number of spots on two faces is ten. Using *Wolfram Alpha*, show the sample space of this experiment in the form of an addition table. Construct a spreadsheet using the RANDBETWEEN function to find an experimental probability of this outcome.

2. Two dice are rolled. Find the theoretical probability that the number of spots on two faces is eight. Using *Wolfram Alpha*, show the sample space of this experiment in the form of an addition table. Construct a spreadsheet using the RANDBETWEEN function to find an experimental probability of this outcome.

3. Two dice are rolled. Find the theoretical probability that the number of spots on two faces is six. Using *Wolfram Alpha*, show the sample space of this experiment in the form of an addition table. Construct a spreadsheet using the RANDBETWEEN function to find an experimental probability of this outcome.

4. A coin changing machine randomly changes a dollar into half dollars, quarters, and dimes. Represent a sample space of this experiment in the form of a table. Assuming that it is equally likely to get any combination of the coins find the probability of having no dimes in a change.

5. A machine changed a half dollar coin into quarters, dimes, and nickels. Assuming that it is equally likely to get any combination of the coins, find the probability that
(i) there are no nickels in the change; (ii) there are no dimes in the change;
(iii) there are no quarters in the change.

6. A machine changed a dollar coin into half dollars, quarters, and dimes. Assuming that it is equally likely to get any combination of the coins, find the probability that

(i) there are no dimes in the change; (ii) there are no quarters in the change; (iii) there are no half dollars in the change?

7. A half dollar is randomly changed into quarters, dimes, and nickels. Draw a sample space for this experiment in the form of a table.

8. A dollar is randomly changed into half dollars, quarters, and dimes. Draw a sample space in the form of a table.

9. Using 10 tiles, construct 5 towers and arrange them from the lowest to the highest. There may be more than one way to do that. Construct all such combinations of 5 towers using the 10 tiles. Record your combinations (the sets of towers). For each set of towers find the mean number of stories, the median number of stories, and the mode number of stories. Describe what you have found. *Andy claimed that he constructed a set of 5 towers out of the 10 tiles. Assuming that it is equally likely to construct any set of 5 towers out of the 10 tiles, what is the probability that Andy's towers are all the same size?*

10. Using 8 tiles, construct 4 towers and arrange them from the lowest to the highest. There may be more than one way to do that. Record your combinations (the sets of towers). For each set of towers find the mean number of stories, the median number of stories, and the mode number of stories. Describe what you have found. *Ann claimed that she constructed a set of 4 towers out of the 8 tiles. Assuming that it is equally likely to construct any set of 4 towers out of the 8 tiles, what is the probability that Ann's towers are all the same size?*

11. Given data set {1, 3, 6, 10, 10, 13, 15, 15, 15, 16} determine the mean, median, mode, and range.

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