# FINDING COMPLEX ROOTS

## Can You Trust Your Calculator?

**John W. Watson and Barbara A. Ciesla**

The use of technology raises interesting questions and presents unique opportunities for in-depth classroom investigations. These opportunities can open a whole new world of understanding to our students. Trying to understand a calculator answer that may be different from the answer in the back of the text or what the class determines is the "correct" answer can be a powerful motivator. Technology also occasionally gives incorrect, or at best highly misleading, information. For example, enter nDeriv(abs(x),x,0) on a TI-83 and it returns a value of 0, in spite of the fact that the derivative of $|x|$ at $x = 0$ does not exist. Technology can also give our students a false sense of familiarity with and understanding of certain concepts. Ask a student to evaluate $3^{\sqrt{2}}$ and you will quickly (assuming their calculator is handy) get an answer of "about 4.73." Persist by asking what the expression means and you are likely to get a blank stare. Some of us attended high school and college without the aid of scientific or graphing calculators. We had the advantage of using calculus texts in which the authors were careful to talk about $a^x$ only in the context of $a > 0$ and $x$ **rational** until, for example, $\ln(x)$ could be defined in terms of an integral, $e^x$ defined as its inverse, and the Intermediate Value theorem invoked to give meaning to $e^x$, when $x$ is irrational.

Then the author(s) defined $a^x$, for $a > 0$ and $x$ any real, by $a^x = e^{x\ln(a)}$. (See, for example, Protter and Morrey 1964.) Add to technology a current high school precalculus text asking students to find *the* value of $(-4i)^{1/10}$ and you have a scenario in which students are likely to be confused (whether they realize it or not) about the properties of complex numbers.

The purpose of this article is to investigate a specific instance when high school students found an answer that coincided with the textbook answer but was different from the one given by their calculator. We will explain the reason for the difference and continue the discussion to extend the definition of powers of numbers to powers that involve complex numbers as the base or exponent. The fact that many calculators are now capable of computing a power such as $(3 + 2i)^{(-5+7i)}$ leads us to believe that we should be ready to explain to our students the basis for such computations.

Here is the difficulty encountered by students in Barbara Ciesla's high school precalculus class when they found that their text and the TI-83 Plus differed on what was the principal root of a number.

"For example," they said, "the TI-83 Plus shows that $(8\mathrm{i})^{1/3} = 1.73 + i$ ; and this answer agrees with the text. Also, according to the TI-83 Plus, $(-4\mathrm{i})^{1/10} = 1.13 - 0.18i$. However, the correct answer [i.e., the answer in the back of the book] is $1.02 + 0.52i$. The TI-83 Plus only gives the correct answer when the problem is entered as $(-4)^{1/10} \cdot i^{1/10}$. We can't guess when to enter a problem in a different form to get the right answer. Why is this occurring?"

Here is a unique opportunity to do some exploration into regions where few high school students tread. In addition, many undergraduate mathematics education programs do not require a course in complex variables, leaving any coverage of complex numbers to occur sporadically in the curriculum. Thus, many teachers do not have a ready answer to the question posed.

We will assume without proof that a strictly increasing or strictly decreasing function is one-to-one, and thus invertible. For example, $f: R \to R$ defined by $f(x) = x^3$ has inverse $f^{-1}(x) = x^{1/3}$ (**fig. 1**). Thus, for example, we can speak unambiguously about $(-8)^{1/3}$. Not so with $4^{1/2}$. We must all agree that unless instructed otherwise, $4^{1/2} = 2$, the positive number whose square is 4. Why is such an agreement necessary? Because $f(x) = x^2$ is not a one-to-one function when defined on all real numbers (**fig. 2**). By restricting this function to nonnegative or nonpositive real numbers, however, it is invertible, and its inverse is $f^{-1}(x) = x^{1/2}$. Thus, if we define $f: (-\infty,0] \to R$ by $f(x) = x^2$, then $f$ is invertible and $f^{-1}: [0,\infty) \to (-\infty,0]$ is given by $f^{-1}(x) = x^{1/2}$, where this means the nonpositive number whose square is $x$. Therefore, $f^{-1}(4) = 4^{1/2} = -2$.

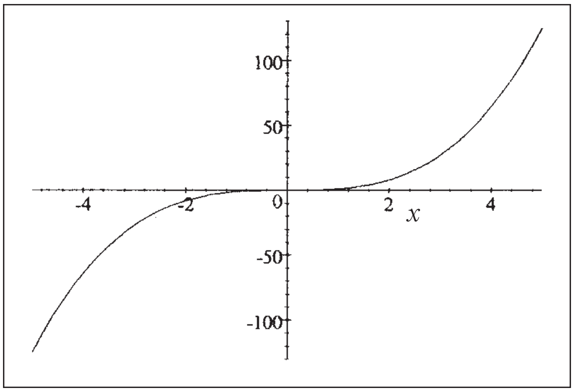


**Fig. 1** $y = x^3$

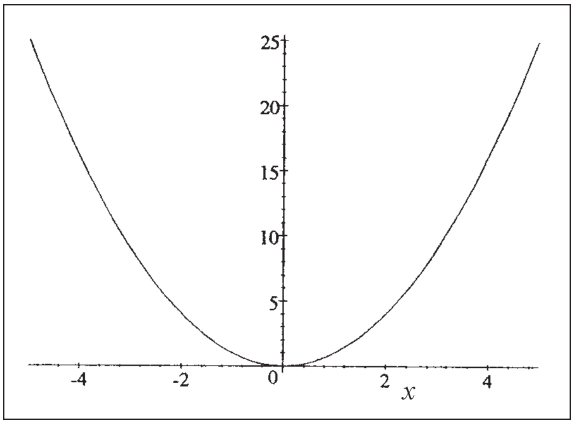


**Fig. 2** $y = x^2$

Before moving on to the more sophisticated task of taking tenth roots of a complex number, we would do well to recall some facts about some important functions in mathematics. The function $\ln(x)$ is a one-to-one function, while the functions $\sin(x)$ and $\cos(x)$ are not (**figs. 3, 4, 5**). We will accept from calculus that $\ln(x)$ is an increasing function, defined for all $x > 0$, and that its inverse $e^y$ for $y$ real, is that unique number such that $\ln(e^y) = y$. You will recall from trigonometry that to define the inverse sine and inverse cosine functions, the domains of $\sin(x)$ and $\cos(x)$ are restricted to $-\pi/2 \le x \le \pi/2$, and $0 \le x \le \pi$, respectively. These two functions, thus restricted, are both one-to-one and their inverse functions exist. Of course, one could also restrict their domains to certain other intervals and define inverses on those intervals. We define $\sin^{-1}(x) = y$ if and only if $\sin(y) = x$ and $-\pi/2 \le y \le \pi/2$. Also, we define $\cos^{-1}(x) = y$ if and only if $\cos(y) = x$ and $0 \le y \le \pi$.

A complex number $z = a + bi$ can be written $z = r(\cos(\theta) + i\sin(\theta))$, where $r = \sqrt{a^2 + b^2}$ and $\theta$ is the angle, $0 \le \theta < 2\pi$, formed by the positive $x$-axis and the ray extending from the origin through $z$ (**fig. 6**). Students appreciate the labor-saving De Moivre's theorem, $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$,

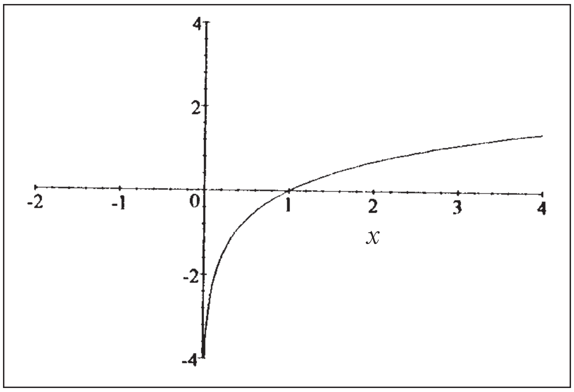


**Fig. 3** ln(*x*)

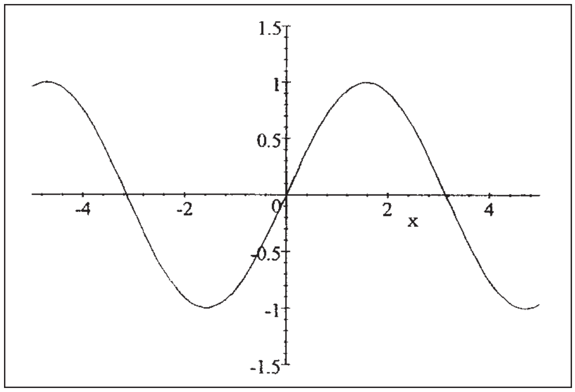


**Fig. 4** sin(*x*)

and its corollary for taking roots of complex numbers:

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right),$$

$$k = 0, \ldots, n-1$$

This theorem and corollary lead one to believe that there are $n$ distinct $n$th roots of a complex number. Should we attempt to designate one of them as the "principal" $n$th root? If so, which one? Even the calculators (actually, the calculator programmers) seem to have trouble with this question. The TI-83 Plus gives $-2 = (-8)^{1/3}$, whether in real or "$a + bi$" mode. However, when entered equivalently as $(-8 + 0 \cdot i)^{1/3}$, the calculator gives an answer of $1 + 1.732i$, whether in real or "$a + bi$" mode. The TI-89, on the other hand, gives $-2 = (-8)^{1/3} = (-8 + 0 \cdot i)^{1/3}$ in real mode, and $1 + \sqrt{3}i = (-8)^{1/3} = (-8 + 0 \cdot i)^{1/3}$ in rectangular mode.

Let's get back to the original question about $(-4i)^{1/10}$. Since $-4i = 4(\cos(3\pi/2) + i\sin(3\pi/2))$,

$$(-4i)^{\frac{1}{10}} = 4^{\frac{1}{10}}\left(\cos\left(\frac{\frac{3\pi}{2}+2k\pi}{10}\right)+i\sin\left(\frac{\frac{3\pi}{2}+2k\pi}{10}\right)\right).$$

Using $k = 0$ (a natural choice for the "principal" root), one obtains an answer of $1.02 + 0.52i$, the answer that the students obtained and that coincides with the answer in the text. Why does the calculator return a different answer? At first one might assume a different choice for $k$ was invoked. However, the answer lies not with the choice of $k$ but with the choice made earlier about our restrictions on the value of $\theta$. What if we wrote $-4i = 4(\cos(-\pi/2) + i\sin(-\pi/2))$? Perhaps the TI-83 Plus chooses $\theta$ such that $-\pi < \theta \le \pi$ instead of $0 \le \theta < 2\pi$. Now we have

$$(-4i)^{\frac{1}{10}} = 4^{\frac{1}{10}}\left(\cos\left(\frac{\frac{-\pi}{2}+2k\pi}{10}\right)+i\sin\left(\frac{\frac{-\pi}{2}+2k\pi}{10}\right)\right).$$

Again using the value $k = 0$, we obtain $1.13 - 0.18i$, which is the answer the TI-83 Plus gives.

Now that we have a likely explanation of why the TI-83 Plus returned this answer, what about writing

$$(-4i)^{\frac{1}{10}} = (-4)^{\frac{1}{10}} \cdot (i)^{\frac{1}{10}}?$$

By using $\theta = \pi$ for $-4$, $\theta = \pi/2$ for $i$, and $k = 0$, one obtains

$$\begin{aligned}
&(-4)^{\frac{1}{10}} \cdot (i)^{\frac{1}{10}} \\
&= 4^{\frac{1}{10}}\left(\cos\left(\frac{\pi}{10}\right)+i\sin\left(\frac{\pi}{10}\right)\right) \\
&\quad \cdot 1^{\frac{1}{10}}\left(\cos\left(\frac{\pi}{20}\right)+i\sin\left(\frac{\pi}{20}\right)\right) \\
&\approx 1.02 + 052i,
\end{aligned}$$

the answer the students got, and the answer the TI-83 Plus gives for

$$(-4)^{\frac{1}{10}} \cdot (i)^{\frac{1}{10}}.$$

So now we have another dilemma. Why does the TI-83 Plus violate the rule $(ab)^p = a^p b^p$? Or does it?

We begin our investigation of this question by asking the following: "What does $z^w$ mean when $z$ and $w$ are complex numbers?" For example, what does $(1 + i)^{2i}$ mean; and if it is meaningful, can it be written in $a + bi$ form?

Let's start with a function with which we are familiar and determine whether its domain can be extended to the complex numbers. First we will define $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. This might seem a bit strange the first time one encounters it, but one motivation for this definition can be seen by remembering some series from calculus:

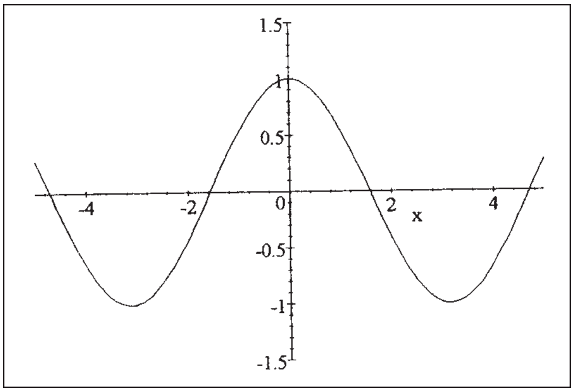


**Fig. 5** cos($x$)

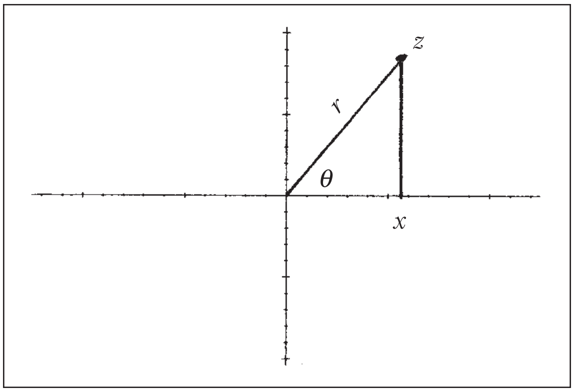


**Fig. 6** z = $r(\cos(\theta) + i\sin(\theta))$

$$e^{\theta} = 1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\cdots+\frac{\theta^n}{n!}+\cdots=\sum_{n=0}^{\infty}\frac{\theta^n}{n!}$$
$$\sin(\theta)=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}$$
$$\cos(\theta)=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n}}{(2n)!}$$

Although the following is not legitimate without a treatment of convergence of complex series, notice that

$$\begin{aligned}&\cos(\theta)+i\sin(\theta)\\&=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots\right)\\&=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\cdots\end{aligned}$$

This last expression, assuming it makes sense, appears to be $e^{i\theta}$ in series form.

We have what seems to be a reasonable definition for $e^{i\theta}$, and we can now define $e^z$ for any complex number $z = a + bi$ by

$$\begin{aligned}e^z &= e^{a+bi}\\&= e^a e^{bi}\\&= e^a(\cos(b) + i(\sin(b)).\end{aligned}$$

Notice that this is an extension of the function $e^x : R \to R$ because if $z$ is real, $z = a + 0 \cdot i$, so $e^z = e^a(\cos(0) + i\sin(0)) = e^a$.

The next fact we should note about our newly defined function is that, unlike its restriction, $e^x$ on $R$, $e^z$ is not one-to-one. In fact it is periodic, as the following shows:

$$\begin{aligned}e^{z+2\pi i} &= e^{a+bi+2\pi i}\\&= e^{a+(b+2\pi)i}\\&= e^a(\cos(b+2\pi)+i\sin(b+2\pi))\\&= e^a(\cos(b)+i\sin(b))\\&= e^z\end{aligned}$$

In our journey toward trying to make sense of $z^w$, let's now remember that for real numbers $a$ and $x$, with $a > 0$, $a^x = e^{x\ln(a)}$. In fact, this is a definition of $a^x$ for $x$ irrational. Might a similar definition hold for complex numbers? If so, we must first define a logarithm of complex numbers, a task that is complicated by the fact that the function $e^z$ is not one-to-one.

We will leave it to the reader to justify the following, or consult Marsden and Hoffman (1999), for details.

1. $e^z e^w = e^{z+w}$ for all complex numbers $z$ and $w$.
2. $e^z = 1$ if and only if $z = 2n\pi i$ where $n$ is an integer.
3. Define $A = \{a + bi \mid -\pi < b \le \pi\}$ and denote the set of complex numbers excluding 0 by $C\backslash\{0\}$. Then the function $f(z) = e^z$ is a one-to-one mapping of $A$ onto $C\backslash\{0\}$ (**fig. 7**). More generally, if we define $A_y = \{a + bi \mid y < b \le y + 2\pi\}$, then $f(z) = e^z$ is a one-to-one mapping of $A_y$ onto $C\backslash\{0\}$.

The reader is encouraged to investigate the images of the lines $y = \pi$, $y = \pi/4$, $y = 0$, and $y = -\pi/4$ under the mapping $e^z$. Understanding this, and the image of the line $y = \theta$, for $-\pi < \theta \le \pi$, will lead to a clear understanding of why $e^z$ is a one-to-one mapping of $A$ onto $C\backslash\{0\}$.)

We can now define a "branch" of the logarithm function for complex numbers. If $z = a + bi$ is a nonzero complex number, then $\log_A(z)$ is that unique number such that $e^{\log_A(z)} = z$. In fact, $\log_A(z) = \ln(|z|) + i\theta$, where $\theta$ is the argument of $z$ satisfying $-\pi < \theta \le \pi$. This is verified by observing that

$$\begin{aligned}e^{\log_A(z)} &= e^{\ln(|z|)+i\theta}\\&= e^{\ln(|z|)}e^{i\theta}\\&=|z|(\cos(\theta)+i\sin(\theta))\\&=\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}+i\frac{b}{\sqrt{a^2+b^2}}\right)\\&= a+bi\\&= z\end{aligned}$$

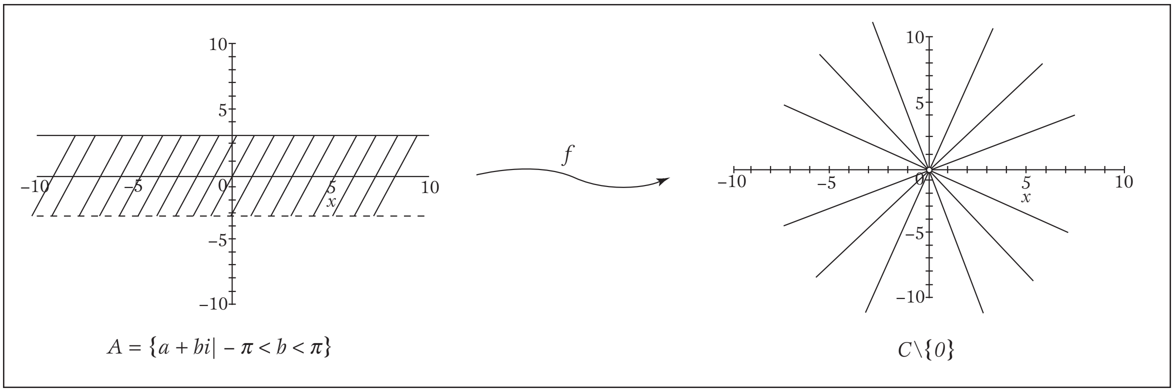


**Fig. 7** $f(z) = e^z$

and

$$\begin{aligned}\log_A(e^z) &= \log_A(e^a e^{bi})\\ &= \ln(e^a) + i\arg(e^a e^{bi})\\ &= a + bi\\ &= z.\end{aligned}$$

We can now define $z^w$, for any complex numbers $z$ and $w$, by $z^w = e^{w \log_A(z)}$. For example,

$$\begin{aligned}(1+i)^{2i} &= e^{2i\log_A(1+i)}\\ &= e^{2i\left(\ln(\sqrt{2})+i\frac{\pi}{4}\right)}\\ &= e^{\frac{-\pi}{2}+i\ln(2)}\\ &= e^{\frac{-\pi}{2}}(\cos(\ln 2) + i\sin(\ln 2)).\end{aligned}$$

This result is approximately equal to 0.160 + 0.133$i$, which is the same answer the TI-83 Plus returns. Suppose, however, we were using the branch of the log function defined by using

$$A_{\frac{\pi}{2}} = \left\{a + bi \mid \frac{\pi}{2} < \theta \le \frac{3\pi}{2}\right\}.$$

In this case,

$$\begin{aligned}(1+i)^{2i} &= e^{2i\log_{A_{\pi/2}}(1+i)}\\ &= e^{2i\left(\ln\sqrt{2}+\frac{9\pi}{4}i\right)}\\ &= e^{\frac{-9\pi}{4}+i\ln(2)}\\ &= e^{\frac{-9\pi}{4}}(\cos(\ln 2) + i\sin(\ln 2)).\end{aligned}$$

This result is approximately 0.000000577 + 0.000000463$i$. Thus, as one can see, the value of $z^w$ depends, in general, upon the branch of the log function being used. This should not be too surprising, because when using the corollary to DeMoivre's theorem to find the values of $z^{1/n}$, we find that there are $n$ solutions. In general, the following can be shown, and the reader is referred to Marsden and Hoffman (1999, p. 33) for details.

THEOREM. *Let $z$ and $w$ be complex numbers with $z \ne 0$. Then*

1. *$z^w$ is single valued if and only if $w$ is an integer. Single valued means that the value does not depend on the choice of the branch for log.*
2. *If $w$ is a real rational number, and if $w = p/q$ in lowest terms, then $z^w$ has exactly $q$ distinct values, namely, the $q$ $q$th roots of $z^p$.*
3. *If $w$ is real and irrational, or $w$ is not a real number, then $z^w$ has infinitely many values.*

Now let's return to the problem that was so troubling to the precalculus class:

$$\begin{aligned}(-4i)^{\frac{1}{10}} &= e^{\frac{1}{10}\log_A(-4i)}\\ &= e^{\frac{1}{10}\left[\ln(4)+i\left(\frac{-\pi}{2}\right)\right]}\\ &= e^{\ln\left(4^{\frac{1}{10}}\right)-\frac{\pi}{20}i}\\ &= 4^{\frac{1}{10}}\left(\cos\left(\frac{\pi}{20}\right) - i\sin\left(\frac{\pi}{20}\right)\right)\\ &\approx 1.13 - 0.18i\end{aligned}$$

However,

$$\begin{aligned}(-4)^{\frac{1}{10}} &= e^{\frac{1}{10}\log_A(-4)}\\ &= e^{\frac{1}{10}(\ln(4)+i\pi)}\\ &= e^{\ln\left(4^{\frac{1}{10}}\right)}e^{\frac{i\pi}{10}}\\ &= 4^{\frac{1}{10}}\left(\cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right)\right)\end{aligned}$$

and

$$\begin{aligned}i^{\frac{1}{10}} &= e^{\frac{1}{10}\log_A(i)}\\ &= e^{\frac{1}{10}\left(\ln(1)+i\frac{\pi}{2}\right)}\\ &= e^{i\frac{\pi}{20}}\\ &= \cos\left(\frac{\pi}{20}\right) + i\sin\left(\frac{\pi}{20}\right).\end{aligned}$$

Therefore,

$$
\begin{aligned}
&(-4)^{\frac{1}{10}}(i)^{\frac{1}{10}} \\
&= 4^{\frac{1}{10}}\left[\cos\left(\frac{\pi}{10}\right)+i\sin\left(\frac{\pi}{10}\right)\right]\cdot\left[\cos\left(\frac{\pi}{20}\right)+i\sin\left(\frac{\pi}{20}\right)\right] \\
&= 4^{\frac{1}{10}}\left[\cos\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{20}\right)-\sin\left(\frac{\pi}{10}\right)\sin\left(\frac{\pi}{20}\right)\right. \\
&\qquad\left.+i\left(\sin\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{20}\right)+\sin\left(\frac{\pi}{20}\right)\cos\left(\frac{\pi}{10}\right)\right)\right] \\
&= 4^{\frac{1}{10}}\left(\cos\left(\frac{3\pi}{20}\right)+i\sin\left(\frac{3\pi}{20}\right)\right) \\
&\approx 1.023+0.521i.
\end{aligned}
$$

So does the TI-83 Plus violate the law $(ab)^p = a^p b^p$? As this example shows, this law of exponents does not apply to complex numbers.

In summary, we would argue that merely showing our students DeMoivre's theorem and its corollary, and then asking them to practice by finding some "principal" root of a complex number, only furthers the misguided idea that mathematics is just a bunch of formulas to be applied to arrive at someone's given answer. Also, students would do well not to take an answer at face value, no matter what its source.

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**Statement of Ownership, Management, and Circulation**

Statement of ownership, management, and circulation (Required by 39 U.S.C. 3685). 1. Publication title: *Mathematics Teacher*. **2.** Publication number: 334-020. 3. Filing date: September 30, 2005. 4. Issue frequency: Monthly, August–December; February–May. 5. Number of issues published annually: 9. 6. Annual subscription price: $32. 7. Complete mailing address of known office of publication: National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 20191-1502, Fairfax County. Contact person: Sandy Berger, (703) 620-9840, ext. 2192. 8. Complete mailing address of headquarters or general business office of publisher: same as #7. 9. Full names and complete mailing addresses of publisher, editor, and managing editor. Publisher: National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 20191-1502. Editor: none. Managing editor: Sandy Berger, 1906 Association Drive, Reston, VA 20191-1502. 10. Owner: National Council of Teachers of Mathematics (nonprofit organization), 501 (c)3, 1906 Association Drive, Reston, VA 20191-1502. 11. Known bondholders, mortgagees, and other security holders owning or holding 1 percent or more of total amount of bonds, mortgages, or other securities: none. 12. Tax status. The purpose, function, and nonprofit status of this organization and the exempt status for federal income tax purposes has not changed during preceding 12 months. 13. Publication title: *Mathematics Teacher*. 14. Issue date for circulation data below: August 2005. 15. Extent and nature of circulation. Average no. copies each issue during preceding 12 months. A. Total number of copies: 34,767. B. Paid and/or requested circulation: (1) paid/requested outside-county mail subscriptions stated on form 3541: 33,303; (2) paid in-county subscriptions stated on form 3541: none; (3) sales through dealers and carriers, street vendors, counter sales, and other non-USPS paid distribution: none; (4) other classes mailed through the USPS: none. C. Total paid and/or requested circulation: 33,303. D. Free distribution by mail: (1) outside-county as stated on form 3541: 1,322; (2) in-county as stated on form 3541: none; (3) other classes mailed through the USPS: none. E. Free distribution outside the mail: none. F. Total free distribution: 1,322. G. Total distribution: 34,625. H. Copies not distributed: 142. I. Total: 34,767. (J) Percent paid and/or requested circulation: 96%. 15. Extent and nature of circulation. No. copies of single issue published nearest to filing date. A. Total number of copies: 33,000. B. Paid and/or requested circulation: (1) paid/requested outside-county mail subscriptions stated on form 3541: 31,524; (2) paid in-county subscriptions stated on form 3541: none; (3) sales through dealers and carriers, street vendors, counter sales, and other non-USPS paid distribution: none; (4) other classes mailed through the USPS: none. C. Total paid and/or requested circulation: 31,524. D. Free distribution by mail: (1) outside-county as stated on form 3541: 750; (2) in-county as stated on form 3541: none; (3) other classes mailed through the USPS: none. E. Free distribution outside the mail: none. F. Total free distribution: 750. G. Total distribution: 32,274. H. Copies not distributed: 726. I. Total: 33,000. (J) Percent paid and/or requested circulation: 98%. 16. Publication of statement of ownership will be printed in the December 2005 issue of this publication. 17. Signature and title of editor, publisher, business manager, or owner: Sandra L. Berger, managing editor, September 30, 2005. I certify that all information furnished on this form is true and complete. I understand that anyone who furnishes false or misleading information on this form or who omits material or information requested on the form may be subject to criminal sanctions (including fines and imprisonment) and/or civil sanctions (including civil penalties).