Tom Cooney, Mathematics, Pedagogy and Secondary Teacher Education, Heinemann, 1996.

Thinking About Being a Mathematics Teacher

Here, we begin by inviting you to explore what mathematics means to you and your views of the teaching and learning of mathematics. We also invite you to consider the way mathematics and the teaching of learning of mathematics are characterized in the NCTM *Standards* and how these perspectives might relate to your professional development. Let us begin by exploring a question that affects all of us—regardless of our career stage.

WHY DID YOU CHOOSE TO BECOME A MATHEMATICS TEACHER?

Reflective Problem 1

Before reading this section, answer the following questions. You may want to write your responses down.

1. Why did/do you want to become a mathematics teacher?
2. If you were to start your education or career over again and you couldn't be a mathematics teacher, what career would you choose? Why?

What influenced you to become a teacher of mathematics? A love of mathematics? A love of working with young people? Financial reasons? Job security? Family circumstances? Do you think your decision was based more on people who influenced you or on circumstantial events in your life? Each of us is attracted to mathematics education for a variety of reasons that depend on our personal biography and circumstances. There is probably not one single reason but rather a collection of reasons and circumstances that influenced our decision.

Think about what people may have influenced you to become a teacher of mathematics. What were their characteristics? What was there about those people that influenced you? Most of us were influenced as young people by parents, community leaders, teachers, coaches, friends, or others whom we admired and with whom we had sustained contact. We admired them for some reason—most likely a certain strength of character, perhaps manifested by warmth and caring, or a certain intellectual curiosity.

We can also ask the mirror-image question about ourselves: Which of *our* characteristics will likely influence the students we teach? How is it that we can create a context within which our students will develop a love of mathematics, curiosity, and a desire to learn? We should never lose sight of the fact that as teachers we will likely influence students in many ways. An important question is, "How can we influence them in a way that honors their growth as thinking and caring individuals?"

To what extent do you think you chose to become a mathematics teacher because of circumstances in your life? Some people chose teaching because of perceived job security or because they feel that teaching fits their particular life-style. To what extent do you think events in your life determined that teaching was the most viable career option for you? Which of the following two factors likely influenced you the most: the subject of mathematics or a desire to work with young people? As you reflect upon your own background, do you remember always being interested in mathematics? Always wanting to help others? To what extent were these factors important (or not) in your decision to become a mathematics teacher?

WHAT COMES TO YOUR MIND WHEN YOU THINK OF MATHEMATICS?

Reflective Problem 2

Consider the following questions. Write down your responses.

1. If someone were to ask me what mathematics is, I would say ______________________.
2. To me, the essence of mathematics is ______________________.
3. The thing that I enjoy most about doing mathematics is ______________________.

Mathematics means many things to many people. For some, mathematics is the study of patterns. Some patterns are simple number patterns that young children can explore. Other patterns are exceedingly complex and are understood by only a few of the world's most renowned mathematicians. While the topics and levels of sophistication may vary greatly, there is nothing about a pattern-seeking conception of mathematics that prevents students of all ages from engaging in a search for relationships, forming conjectures about those relationships, and studying the conditions under which those relationships hold true.

Many see mathematics as a search for abstract relationships that hold "eternal truths." Others see mathematics as an outgrowth of human invention rather than the discovery of what has been there forever. Still others are less interested in the philosophical debate about how mathematics was created and are more interested in the practical side of mathematics. For these latter people, mathematics is seen as a tool for solving practical problems, for example, determining the cost of purchasing a car. Others see mathematics as a kind of puzzle—determining which pieces fit together to make a coherent whole. These people often think of mathematics as a matter of solving unique kinds of problems, perhaps couched in a puzzle format as in the well-known problem of identifying digits associated with each letter in the following arrangement.

```
  S E N D
+ M O R E
M O N E Y
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In *Mathematics, Pedagogy, and Secondary Teacher Education* we try to capture a variety of perspectives about what constitutes mathematics. The following five problems are examples of those presented by the various authors. Do not solve them now but rank-order them in terms of your interest level in solving them.

1. The first two numbers that have three divisors are 4 and 9. These numbers are the first two squares (beyond 1). A reasonable conjecture would be that all squares greater than 1 have three divisors. Is that true? Investigate.
2. ABCD is a unit square. E, F, G, and H are midpoints (Figure 1). What percent of ABCD is shaded?

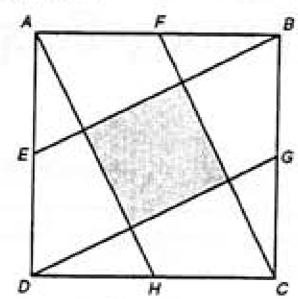


Figure 1

3. Suppose a 20 cm by 20 cm sheet of construction paper had its corners cut out so that it could be folded to make a box (Figure 2). What size corners should be cut out in order to maximize the volume of the box?

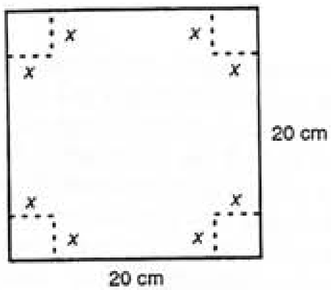


Figure 2

4. A single domino shows a pair of numbers both between 0 and 6. A complete set of dominoes consists of a certain number of dominoes such that every pair of this kind occurs exactly once. How many dominoes are in a set?
5. Prices for pizzas at Gumpy's Pizza and at Pizza Shack are shown below. What advice would you give your friends about purchasing pizza from the two pizza restaurants?

Gumpy's Pizza		*Pizza Shack Pizza*	
Diameter	*Cost*	*Diameter*	*Cost*
20 cm	$6.50	25 cm	$7.00
30 cm	$9.00	35 cm	$10.75
40 cm	$12.50	45 cm	$15.50

Compare your ranking with those of other classmates. Are your rankings similar to theirs or not? Do you think your classmates hold a view of mathematics similar to or different from yours?

One of the authors had the following interview with a teacher several years ago.

I: If you were to think of something that is as different from mathematics as you could imagine, what would it be?

T: That's an interesting question. (*Pause*) I guess I would say hard rock music.

I: And why do you say that?

T: Well that kind of music seems so bizarre. It doesn't seem to have any structure—at least none that I can detect or appreciate. On the other hand, mathematics is very structured. Everything fits together. There are no pieces just laying out there not connected to anything else.

Although we might disagree (or not) with the teacher's analysis of hard rock music and the certainty of mathematics, we nevertheless gain a glimpse of what she thinks mathematics is. Notice that thinking of mathematics as a structure is an important part of her view of mathematics.

Exploration 1

Find a willing friend and ask him or her the following question: If you could think of something that is as different from mathematics as possible, what would it be? Explore why he or she picked whatever was picked. Write a one-page report on what you think your friend's view of mathematics is.

Let us now return to Reflective Problem 2. Consider your response and complete the following activity.

Reflective Problem 3

Write a one- to two-page statement of what mathematics means to you. Include your responses to the various questions posed in this section.

WHAT INFLUENCES OUR VIEWS OF MATHEMATICS?

While we enjoy doing mathematics at some level and generally have a positive attitude toward mathematics, there are many people who hold quite a different view of the subject. For some people, studying mathematics creates considerable anxiety and tension. They think mathematics is a subject that is accessible only to the selected few. Indeed, our society often promotes the view that mathematics is something to be avoided. Consider the cartoon in Figure 3.

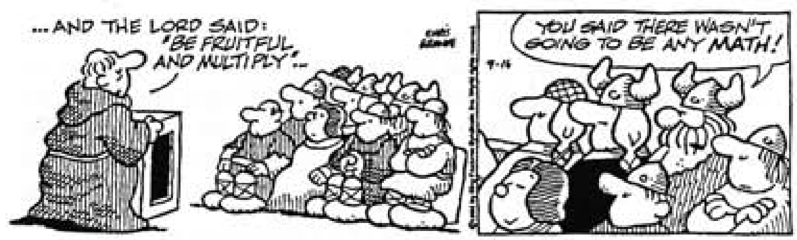


Figure 3

What view of mathematics do you think is communicated in this cartoon?

Furinghetti (1993) has observed that various scenes in movies often depict a certain negativism toward mathematics. For example, she notes that in the film *A Streetcar Named Desire,* Blanche tells her suitor, falsely, that she is a teacher. When asked if she teaches mathematics, Blanche flatly denies that she has anything to do with mathematics, as if such an association would be an affront to her femininity. Furinghetti (1993) also observed that Woody Allen, in *Radio Days*, asks a classmate to recite mathematical formulae to illustrate what a disagreeable character the classmate really is. A number of years ago, a toy company produced a Barbie doll that said, "Math is tough." While the doll was taken off the market, it is easy to see why so many people were offended by a company suggesting that mathematics is a subject inherently difficult for women.

Exploration 2

Identify instances in newspapers, television, or the movies where a particular view about mathematics is conveyed. Describe the situation. Write a brief analysis about what you think the impact might be on others regarding their view of mathematics.

We are often surrounded by circumstances that communicate a certain elitism toward mathematics that is generally counterproductive to helping students see the beauty and joy in doing mathematics. We have all encountered individuals who communicate in a bragging sort of way that they don't know much about mathematics. We are a product of this society. Yet, for the most part, we don't share that view. For us, mathematics is an enjoyable subject, not discounting the fact that we may have had many difficult and trying experiences when learning mathematics. We cannot deny the fact that many of our students' views of mathematics, and perhaps our own as well, are influenced by the culture in which we live. To the extent to which this culture communicates a negative view of mathematics, we are faced with the challenge of communicating to our students that mathematics is a subject accessible to all.

WHAT DO THE NCTM *STANDARDS* SAY ABOUT MATHEMATICS?

The view of mathematics communicated in the NCTM *Standards* places considerable emphasis on various processes in doing mathematics, particularly communication, reasoning, and problem solving. We will consider the nature of each of these processes as addressed in the *Standards*.

MATHEMATICS AS COMMUNICATION

We can think of communication in at least two different ways. First, there is the need to communicate to others what we mean by the mathematics we are using. Second, we can use mathematics to communicate information about the real world. The following activity emphasizes mathematical communication in the sense of using mathematical terms to help another student draw a figure.

Exploration 3

Work in groups of three. One person should be shown Figure 4 or a similarly drawn figure. This person should give directions to a second person who is to draw the figure but who cannot see the figure. The first person cannot see what the second person is drawing and hence is unaware of the result of the directions given. The third person should note what mathematical terms are used by the first person.

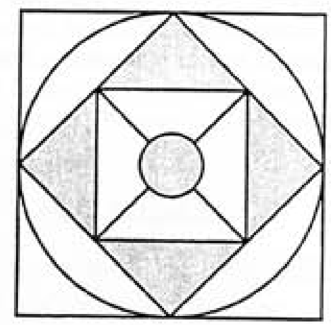


Figure 4

When the first person completes his/her directions, the three participants should discuss the process and observe whether the second person drew the figure correctly or not. The third person should note ways that the mathematical language was used effectively and how it might have been used better, if possible.

Mathematics also provides a means by which we can describe and analyze real-world phenomena.

Exploration 4

Identify several instances in newspapers, television, or the movies where mathematics is used to communicate information. Write a short essay on the role mathematics can play in communicating this information to the general public.

MATHEMATICS AS REASONING

Mathematics by its very nature involves different kinds of reasoning processes. The NCTM *Standards* emphasize four different kinds of reasoning: inductive, deductive, proportional, and spatial. *Inductive reasoning* involves a process by which we recognize what is common to a set of examples and then generalize the observed property to a more inclusive set. For example, if we square an odd number and subtract one from the square, we obtain the following examples.

$3^2 - 1 = 8$ $5^2 - 1 = 24$ $7^2 - 1 = 48$ $9^2 - 1 = 80$ $11^2 - 1 = 120$

We can make several observations. First, the resulting numbers are all even. Second, the resulting numbers are all multiples of 8. Third, the numbers all end in 0, 4, or 8. We could, therefore, form the following generalizations.

- The square of an odd number less one is an even number.
- The square of an odd number less one is a multiple of 8.
- The square of an odd number less one results in a number that ends in 0, 4, or 8.

Deductive reasoning can help us establish whether conjectures are true by using logic that proceeds from the general to the specific. Often deductive reasoning fits one of the two following patterns.

Modus Ponens	*Modus Tollens*
If p, then q p Therefore q	If p, then q not q Therefore not p
If a figure is a square, then the figure is a parallelogram ABCD is a square Therefore ABCD is a parallelogram	If a figure is a square, then the figure is a parallelogram ABCD is not a parallelogram Therefore ABCD is not a square

Deductive reasoning also occurs in the form of a series of deductions called the *chain rule* as indicated below.

$p \rightarrow q$ (read "p implies q")
$q \rightarrow r$
Therefore $p \rightarrow r$

The following example illustrates this kind of reasoning.

If ABCD is a square, then ABCD is a rectangle.
If ABCD is a rectangle, then ABCD is a parallelogram.
Therefore if ABCD is a square, then ABCD is a parallelogram.

Let us return to the second of the three conjectures stated above. We can establish it by using deductive reasoning.

If n is an odd number, then it can be expressed as $n = 2k + 1$, where k is some integer.

Therefore $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$. If we subtract one, we have $4k^2 + 4k$, which can be factored $4k(k + 1)$. Since either k or $k + 1$ must be even, we can see that the expression $4k^2 + 4k$ must be a multiple of eight since it has a factor of four and a factor of two.

You may want to consider how the first and third conjectures can be established using deductive reasoning.

It does not follow that all conclusions based on inductive reasoning are true. We saw that the first two square numbers (beyond 1), 4 and 9, have three factors. Twenty-five also has exactly 3 factors: 1, 5, and 25. We might conjecture that all square numbers (beyond 1) have 3 factors. But 16 has 5 factors! Neither is it the case that when a generalization is reached using inductive reasoning, the truth of the generalization can be established. For example, Goldbach (1690–1764) observed that any even number greater than 2 can be represented as the sum of two prime numbers. Observe that $4 = 2 + 2$, $20 = 17 + 3$, and $52 = 47 + 5$. This conjecture is called *Goldbach's conjecture*. While computers have yet to find a counterexample, neither has anyone created a proof for Goldbach's conjecture. Similarly, the seventeenth-century mathematician Pierre de Fermat (1608–1665) stated what became known as "Fermat's Last Theorem," which can be stated as follows:

The equation $a^n + b^n = c^n$ is not solvable in integers for any $n > 2$.

While this theorem seemed to be true (no counterexamples could be found), a proof for the theorem did not appear until 1994. This proof is still being studied by mathematicians.

Another kind of reasoning discussed in the NCTM *Standards* is that of *proportional reasoning*. Two quantities vary proportionally when, as their corresponding values increase or decrease, the ratios of the two quantities are always equivalent. If, for example, x and y vary proportionally with x as the independent variable, then there is a multiplicative relationship between x and y such that for each value of x, $y = kx$, k being a constant. Does it follow that if $y = x + 1$, then x and y vary proportionally? Would $(y - 1)$ and x vary proportionally? A second characteristic of two quantities that vary proportionally is that for every unit change in x, there is a constant change in y. Proportional reasoning can be used to solve rather trivial problems like, "If 3 tickets to a movie cost \$12, how much do 12 tickets cost?" A more complex problem might be, "If a 4" by 6" photograph is enlarged to fit an 8" by 10" frame, does the photo need to be cropped? If so, what part of the picture would have to be cropped?" Without question, proportional reasoning is a central part of mathematical thinking.

A fourth kind of reasoning highlighted in the NCTM *Standards* is *spatial reasoning*, which involves reasoning with two- or three-dimensional objects. Consider, for example, what kind of cross section of a cube (Figure 5) would result in the following planar figures.

1. a square
2. a rectangle
3. an isosceles triangle
4. an equilateral triangle
5. an isosceles trapezoid

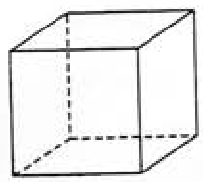


Figure 5

Given these different cross sections, we could ask how their areas could be determined. These tasks require spatial reasoning.

> **Exploration 5**
>
> Suppose the cube above was a unit cube. What would be the area of the largest possible rectangular cross section? The largest possible square cross section? The largest possible triangular cross section?

MATHEMATICS AS PROBLEM SOLVING

Problem solving has always been recognized as an important part of doing mathematics. As mentioned in the NCTM *Standards* there is more to problem solving than just getting answers, e.g., learning and using strategies for solving problems, verifying and interpreting results, reflecting on the solution process, posing new problems, and generalizing results where appropriate. George Polya's classic book *How to Solve It* emphasizes strategies for solving problems and the value gained from looking back on the solution process. The asking of questions like "What happens if?" or "What if not?" provides means of generating problems to be considered that are "cousins" to the initial problem.

Consider the following problem that was shared with one of the authors by a Japanese colleague.

> Jack is in a race with 15 other boys. At the beginning of the race Jack is fifth from the last. At the end of the race he is third. How many boys did he pass?

When people solve this problem, they inevitably draw some sort of representation of the runners and then locate Jack in that representation. But some assumptions need to be made. How many boys are in the race? What do we mean by fifth from the last? What does it mean to finish third? If Jack passes one boy, and then that boy passes Jack, and then finally Jack passes that boy again, is that one pass or two passes for Jack? Does it matter whether Jack passes the two boys who are initially first and second and consequently other boys finish first and second in the race or whether he never passes the two boys who initially lead the race? We might say that while the problem seems simple enough, it is hopelessly vague. Yet, there is much here to be valued. When mathematics is applied to real-world problems, certain assumptions must be made about the context or conditions of the problem being addressed. For example, mathematical models for weather predictions make certain assumptions about climatic conditions. To the extent that those assumptions are correct, the mathematical models can predict weather rather accurately. But when some event occurs that was not accounted for in the model, the prediction goes awry—the model failed to account for all of the relevant factors. The racing problem invites students to consider assumptions necessary to solve the problem.

Another facet of problem solving is developing general solutions from specifically solved problems. Suppose we made the following representation for the race with Jack and assume that he can pass a person only once and that he does not pass the initial two leaders in the race. Jack's assumed position is marked.

Beginning of race: $B_1\ B_2\ B_3\ B_4\ B_5\ B_6\ B_7\ B_8\ B_9\ B_{10}\ B_{11}\ \underline{\mathbf{B_{12}}}\ B_{13}\ B_{14}\ B_{15}\ B_{16}$

End of race: $B_1\ B_2\ \underline{\mathbf{B_{12}}}\ B_3\ B_4\ B_5\ B_6\ B_7\ B_8\ B_9\ B_{10}\ B_{11}\ B_{13}\ B_{14}\ B_{15}\ B_{16}$

Under these assumptions, we can see that Jack passes 9 other boys (B_3 through B_{11} inclusive). The question then becomes, how can we determine that he passes 9 boys without drawing the representations, that is, using only the numbers in the problem? What symbolic representations would we need? This might be important if we were trying to solve the following problem where drawing a representation for the number of girls in the race would not be a reasonable approach.

Exploration 6

Jackie was in a race with 200 other girls. At the beginning of the race she was 50th from last. At the end of the race she was 10th. How many other girls did she pass?

1. Identify other assumptions that could be made in Jack's race and determine what the solutions would be under these assumptions.
2. How would these assumptions affect the solutions to the problem with Jackie?
3. What conditions in the problems could be changed to create other types of problems?

MATHEMATICS AS CONNECTIONS

An important emphasis in the NCTM *Standards* is that of *connections.* In general connections are of two types. First, there are the connections between mathematics and real-world situations. The chapter on mathematical modeling emphasizes this type of connection. In a more pervasive way, this type of mathematical connection can help us think quantitatively as we make decisions that affect our lives. The second type of connection exists within mathematics itself and is rooted in the structure of mathematics.

With respect to the first type of connection, suppose we were operating a flower shop and we wanted to determine what price for flowers and plants would give us a reasonable return on our investment. What factors should we consider and what mathematics would be involved? Clearly we must sell them for a price greater than what we paid for them and yet not so high that everyone is discouraged from buying them. But how do we find a reasonable price between these two extremes? What are the factors to be considered? The cost of the plants and flowers for the store owners? The cost of renting or buying the shop? The volume of the business—how many plants or flowers can be handled in a given year? What the competition sells the plants or flowers for? The salaries for the shop owners and for other employees? Does it make a difference whether special days, e.g., Valentine's Day, come on a weekend or during the week? While experience and trial and error may be helpful in determining prices, there are a number of factors that need to be considered and modeled in order to determine prices that return a reasonable profit.

Exploration 7

Contact owners of a small business who sell a particular product. Ask them what factors contribute to their determination of the prices and what factors seem to influence sales. Write a brief report on what you found and what role you see mathematics plays (or could play) in determining a fair price.

Mathematical connections also include connections within mathematics—the second type of connection. These kinds of connections, for example, can be between algebraic and geometric relationships or between various theorems in the same mathematical domain. Consider the relationships between the following situations.

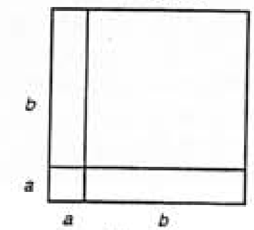
The expansion of $(a+b)^2$	The area of a square with the dissection shown in Figure 6.  Figure 6
The diagonal of a square	The diagonal of a cube
The area of a regular polygon	The area of a circle
The graph of $f(x) = e^x$	The graph of $f(x) = \ln x$

Figure 6

WHAT VIEW DO YOU HAVE ABOUT THE TEACHING AND LEARNING OF MATHEMATICS?

In a previous section you were asked to consider your view of mathematics. The following activity invites you to consider your view of teaching mathematics.

Reflective Problem 4

Consider analogies with the following possibilities and decide which one(s) best fit your notion of what it means to be a mathematics teacher. Provide a rationale as to why you made the selection that you did.

newscaster	orchestra conductor	physician
missionary	gardener	engineer
social worker	entertainer	coach

How does your selection and rationale compare with those of your classmates?

As the above activity suggests, there are many facets to the teaching of mathematics. Indeed, perhaps each of us can identify with some aspect of all of the possible selections.

Reasons for teaching mathematics vary from teacher to teacher. Some enjoy seeing the sparkle in students' eyes when they comprehend something they had not previously understood or the thrill of helping students master a subject that they had previously considered "unlearnable." Others like to present problems and engage themselves and their students in solving problems to which neither teacher nor student knows the answer. Still others may teach mathematics because they feel that learning mathematics teaches students to master details and to become disciplined in their approach to learning.

There are many facets to becoming an effective teacher of mathematics. The NCTM *Professional Standards for Teaching Mathematics* (1991) emphasizes the importance of posing tasks that are based on sound and significant mathematics, posing questions and tasks that elicit, engage, and challenge each student's thinking, listening to students' ideas, enabling students to investigate mathematical ideas and to validate their own mathematical reasoning. In this section we will consider these emphases as they relate to 1. the teaching of mathematics for *all* students, 2. infusing technology into classroom teaching, and 3. assessing students' understanding of mathematics.

THE NOTION OF CULTURAL DIVERSITY

The notion of cultural diversity includes issues of gender, race and ethnicity, and cultural backgrounds of students. The NCTM *Standards* emphasize that tasks and means of teaching mathematics should provide contexts in which all students are encouraged to learn. In many ways, often implicitly and unknowingly, we communicate a certain orientation toward mathematics through the

use of our language and methods of teaching mathematics. Damarin (1990), for example, notes that terms such as *mastery*, mathematical *power*, *attack* problems, apply *strategies*, and *torpedo* generalizations convey a certain aggressiveness that is sometimes offensive to women. Damarin poses the following questions for our consideration as we reflect on our teaching of mathematics.

> How deeply rooted in competition is mathematics instruction?
>
> How can students be encouraged to use "sharing time" to share their own mathematical problems of the economics of allowances, the allotment of time to activities, or quantitative goal setting?
>
> What are some good problems involving quadratic equations that do not involve trajectories (which Damarin tends to associate with bullets and bombs)? (146)

Damarin's purpose is to raise our consciousness in how we talk about and teach mathematics so that we communicate that mathematics is for all students. A key point made by Damarin (1990) is that the means by which we teach mathematics may be biased in favor of students who thrive on competition and verbal learning—characteristics that may work against women and minorities.

The culture of the classroom can be an important determiner of how and whether students learn mathematics. With respect to the teaching of African Americans, Stiff (1990) writes, "The emphasis on making connections, mathematical communication, and cooperative learning, for example, should permit teachers to give a greater sense of security to all students but will make it equally important that teachers understand and value the elements of the culture that make African-Americans unique from others" (156). Urie Treisman, for example, was able to affect dramatically the performance of African-American students in their calculus course at the University of California at Berkeley by explicitly teaching them strategies of cooperation. These strategies were part of the student culture of his Asian pupils but were assumed to be a form of cheating by African Americans in the course. A key component of Treisman's success was the creation of a culture that expected success. The popular film *Stand and Deliver*, which features the teaching of Jaime Escalante, also emphasizes how the creation of a success-oriented classroom produces (Hispanic) students who make significant progress in learning mathematics.

Cuevas (1990) offers specific suggestions for teachers who are dealing with immigrant students or students whose first language is not English. Included are avoiding idioms or language that is particularly couched in the predominant language, being aware of the students' cultural backgrounds when developing mathematical ideas, and providing contexts in which students can share ideas with one another before making presentations to the entire class.

Each of these authors, in different ways, is raising the issue about the role of language in the teaching and learning of mathematics. Students whose backgrounds are not from the dominant culture will likely experience difficulties in learning mathematics if the medium by which they learn is primarily verbal based on the language of the dominant culture. While language will always be an important part of teaching mathematics, it need not be the only means by which mathematics is taught. What is sometimes missed, however, is that the medium by which mathematics is learned may have a lot to do with what and how well it gets learned. The current popularity of using cooperative learning groups stems in part from an emphasis on mathematical communication, but also reflects a desire to better accommodate the needs of diverse student populations.

Another aspect of attending to cultural diversity is the use of problems and situations that have meaning to the students. Even puzzlelike problems that are seemingly "culture free" are more likely to be of interest to students who come from backgrounds where there is a sense of gratification associated with solving puzzles. Damarin's (1990) contention that problems involving quadratics often involve ballistics highlights the fact that some mathematics problems are couched in contexts that may, at the very least, be uninteresting to students, and, at worst, be offensive to them. Given the advancement of women in athletics and in the business world, and of men in such professions as nursing, it is no longer the case that problems necessarily have to be selected to fit stereotyped roles in society. Yet, culture is important to the selection of problems. For example, students from a rural background may not be interested in problems that seemingly address issues common to urban life, and vice versa. While good textbooks often provide an array of problems from different social contexts, teachers should create and use problems that encourage the best possible performance from their students.

Wilson and Padron (1994) provide the following analysis regarding the challenges we face in addressing issues of cultural diversity.

> Schools are trying to provide mathematics for all students through essentially the same practices that have reduced the subject to preparation for the next mathematics class with little relevance to daily living. Students are still placed in tracks, textbooks have essentially the same format and content, curriculum guides are lists of mathematical topics, and teachers' lesson plans consist of exercises to be assigned and textbook pages to be completed. The problem is extensive and can be approached from several perspectives, but we would like to focus on the power of teachers and teacher education programs to make a difference. (47)

The authors show how different cultures approach the same computational problem. For example, they write that students from different countries use the division algorithm in different ways as indicated below (Wilson & Padron, 52).

Venezuela

$$\begin{array}{r|l} 540 & \\ 90 & 45 \\ \cline{2-2} 0 & 12 \end{array}$$

United States

$$\begin{array}{r} 12 \\ 45\overline{)540} \\ \underline{45} \\ 90 \\ \underline{90} \\ 0 \end{array}$$

While 24 ÷ 5 yields the same numerical answer regardless of what algorithm is used, we can envision quite different and culture-dependent answers to the following question.

Create a story in which the answer to the problem 24 ÷ 5 is

1. 4
2. 5
3. 4.8
4. $4\frac{4}{5}$

The stories that people create will reflect their experiences—which, by definition, will be unique to that person. While we might think that 24 ÷ 5 is culture free, certainly the real-world context in which that computation has meaning is anything but culture free.

Exploration 8

Identify a partner and debate the following issue:

Mathematics is culture free. We should not be concerned about the nature of problems students solve so long as they are doing significant mathematics.

Flip a coin. If it comes up heads, you argue in support of the above position while your partner argues to the contrary. If it comes up tails, you argue to the contrary while your partner argues in support of the position.

Carry on the debate for about 10 minutes. At the end of that time, stop the debate and list the arguments made by each person. Compare your and your partner's list with those from other pairs of debaters.

In the following activity, consider the different types of problems that are often found in textbooks. The purpose of the activity is to consider the possibility that the problems students solve may carry with them certain biases in terms of students' interests. The argument can be made that students should encounter a wide range of problems to enable them to appreciate cultures that are different from their own. Would you agree or disagree with such an argument?

Exploration 9

Work in a group of four students and consider the following problems. Analyze the problems in terms of their potential interest level for students in terms of gender, socioeconomic status, and cultural background. Compare your group's analysis with critiques from other groups in the class. (You need not solve the problems.)

1. Kathy has to cut the lawn, which approximates a square 80 feet by 80 feet. She wants to determine whether she should cut the lawn back and forth in vertical strips, whether she should cut the lawn in strips parallel to the diagonal of the square, or cut it in a spiral path. Which path would you recommend to Kathy and why?
2. Andy needs to fertilize a plot of land that measures 150 feet by 200 feet. How much will it cost him if 10 pounds of fertilizer covers 1,000 square feet and each 10-pound bag costs $10?
3. A city transit system is going to increase fares 10 cents per trip. If someone makes two trips a day, how much more will it cost the person over a year if he uses the city transportation 5 times a week, 50 weeks a year?
4. Teresa's free throw shooting average is 70%. If she plays in two games over the weekend and makes 15 out of 21 free throws, has she done better or worse than her average? Justify your answer.
5. Indiana Jones is traveling down a river filled with piranhas in a Brazilian rain forest. When he is going downstream he can cover 1 mile in three minutes. When his boat goes upstream, it takes 15 minutes to cover the mile. What is the speed of the current?
6. The weight of a body is inversely proportional to the square of its distance from the center of the earth. If a man weighs 180 pounds on the earth's surface, what will he weigh 200 miles above the earth? Assume the radius of the earth is 4,000 miles.
7. If a number is increased by 4 times its reciprocal, the sum is $4\frac{1}{6}$. Find the number.
8. If there are 70 tennis players who enter a tournament and play single elimination, how many matches will be played in the tournament?

THE ROLE OF TECHNOLOGY IN TEACHING

There can be little question that technology is playing an increasingly important role in the teaching of mathematics. This raises serious questions about what mathematics should be taught given our capacity to solve much more computationally difficult problems. Still, it seems likely that in the foreseeable future some skills will remain important to the learning of secondary school mathematics. Technology can facilitate the solving of certain problems that might be difficult to solve using standard, algebraic means. For example, graphing calculators can enable us to solve equations that might not be solvable by means usually available to secondary students. Too, technology can give us a different perspective on traditional topics. Consider, for example, the solving of equations such as $x^3 - x = 20$ or $2(x - 5) = 11 - x$. The cubic equation can be solved using graphing calculators in which two functions are graphed ($f(x) = 20$ and $f(x) = x^3 - x$), the solution designated by their point(s) of intersection. The linear equation can be easily solved by traditional algebraic means. But it can also be solved by considering the intersection of the graphs of two functions, $f(x) = 2(x - 5)$ and $f(x) = 11 - x$. Such a solution process provides students with not only a geometric interpretation of solving linear equations (or equations more generally), but also a context for students to develop a broader understanding of what it means to solve equations. Similarly, we could use spreadsheets to solve equations by generating values for each member of the original equation. By using successive approximations, if necessary, we can determine the values for which the expressions $2(x - 5)$ and $11 - x$ assume the same value.

Exploration 10

Solve the following equations in three ways (if possible): using standard algebraic techniques, using a graphing calculator, and using a spreadsheet. (You may need to approximate some solutions.)

1. $-3x + 5 = 4(2 - x)$
2. $3^x = 135$
3. $\sin x = .5x$
4. $x^2 - 5 = |x|$

THE IMPORTANCE OF ASSESSMENT

The NCTM *Standards* emphasize reform not only in teaching but also in the means by which we assess students' learning. The issue of assessment in the reform process is of major importance in mathematics education today. The fact that the National Council of Teachers of Mathematics developed an entire set of standards on assessment is in itself indicative of the importance being given to assessment. In some sense our instruction and what we assess is dependent on what we think mathematics is and what we consider our role as teachers of mathematics to be. If we think mathematics is essentially a matter of acquiring skills and computing, then our tests will likely engage students in doing a large number of skills or computations. On the other hand, if we think our role as teachers is to help students understand the structure of mathematics, then our tests will likely consist of problems that require students to use various forms of deductive reasoning—as, for example, when we ask students to write proofs in geometry. Still other teachers may think that their role as teachers is to help students see how mathematics is a tool for solving real-world problems. Such a view of mathematics and of the teaching of mathematics will lead to test questions couched in real-world contexts.

The evidence is mounting that our beliefs and attitudes toward mathematics influence how we teach mathematics and how we assess students' learning of mathematics. (See, for example, Thompson, 1992.) While it is unclear whether our instruction drives what we test or whether, conversely, what we test drives what we teach, in the practical world of the classroom instruction and assessment are integrally related. There is no shortage of references on assessment in mathematics education that provide suggestions on how to create interesting items for students, how to use portfolios as a means of assessing student growth, and how we can provide contexts that encourage students to demonstrate their full understanding of the mathematics in question. (See, for example, NCTM (1991b), MSEB (1993), and the 1990 and 1993 NCTM yearbooks.) These various assessment techniques are too comprehensive to address here. One issue, however, is essential to the notion of integrating content and pedagogy, namely, the use of open-ended questions.

Open-ended questions are designed to encourage students to demonstrate their reasoning processes, to give evidence of their depth of understanding of mathematics, and to provide a context for assessing their ability to communicate mathematically. We should be clear that having a depth of understanding is quite a different matter from doing a difficult problem. For example, multiplying a 9-digit number by a 9-digit number is difficult in the sense that rarely would a person get the correct answer given the number of possible computational errors involved. Yet few would maintain that such an item assesses a deep and thorough understanding of multiplication. Assessing a "deep understanding" of mathematics entails the use of questions that reveal how students see mathematics as a connected whole and not as isolated bits of information. Further, it requires a demonstration of reasoning and the ability to communicate one's thinking. For example, we can ask students, "Three fifths of what number is 15?" but we can't be sure how connected their knowledge is to the more general topic of equivalent fractions. The simplest

way to gain this information is to require students to "explain your reasoning." Other techniques that teachers find useful is to consider general questions such as the following.

> What's wrong with this?
> One person says this and another person says that (a contradiction). Who is correct and why?

The following questions illustrate these two types of general questions.

1. Without referring to a textbook or another person, how could you convince someone that the following statement is false?
   The general solution to $ax^2 + bx + c = 0$, $a \neq 0$ is
   $$x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$
2. Fred maintains that as the perimeters of rectangles increase, their areas also increase. Alisha claims Fred is wrong. Who is correct and why?

Assessment should provide a wide range of contexts for students to demonstrate their mathematical understanding. Problems that have only single number answers are not likely, by themselves, to provide that range.

Some teachers find the incorporation of open-ended questions or items into their teaching a real challenge. A reasonable first step toward using such items is to engage students in a kind of "Jeopardy" game show. That is, instead of asking for an answer, have students imagine what a possible question might be if they are given a particular answer. Another is to make use of real or imagined disagreements that might occur naturally in class discussions. Below are two efforts in that direction by a teacher who originally had difficulty incorporating open-ended items into her assessment of students' understanding.

"Old" Item	*"New" Item*
Round each of the following to the nearest tenth:	Create a number that would round to be each of the following:
a. 48.128	a. 54.32
b. 1399.92435	b. 853.8
c. 4.2378	c. 91.998
Categorize each of the following numbers as prime or composite: 81, 2, 53, and 111.	Buddy thinks that 91 and 93 are prime because they end in an odd digit. Pete says that Buddy is wrong. Who is correct and why?

The teacher was impressed with the information she gained from her students. From her perspective the more open-ended items gave her an entrée into her students' thinking that had not previously been revealed when using the more traditional items.

An important part of the assessment process is to decide how to score students' responses. Developing a scoring "rubric" is an important part of the assessment process. If students' responses tend to be scored as either "right" or "wrong" then the item is unlikely to assess students' understanding in a deeper or more connected way. Consider the following item given by a secondary teacher and the scoring system he used.

Item:	Is it possible for an equilateral triangle to have a right angle? If so, give an example. If not, why not?
Level One:	Yes. Sides are straight at a right angle.
Level Two:	Yes, as long as all of the sides are the same length.
Level Three:	No, because all sides must be equal.
Level Four:	a. No, because there must be one side of the triangle (hypotenuse) that is longer in a right triangle and an equilateral has all sides the same.
	b. No, all the angles have to be the same and all three have to equal 180 degrees.
Level Five:	a. No, you can't have 3 right angles because the sum of the angles would be 270 degrees and it must equal 180. The angle measures are all the same in an equilateral triangle.
	b. No, because an equilateral triangle has all the same angles. If you had a triangle with 3 right angles, you would have 3/4 of a square but the sides would not connect.

His five-point scale was based on his analysis of what constituted different levels of understanding and on actual students' responses. It is interesting to note that the response classified as Level 4a is based on side length whereas other responses are based on angle measure. How would you respond to this student? How would you have scored the different responses?

It is important to remember that what we assess communicates to students what we value and, in all likelihood, what they will learn. Our tests, quizzes, examinations, and other means of grading students communicate to them what we believe mathematics to be. In a real sense, it defines what they ought to be doing mathematically.

Exploration 11

Analyze the following problems in terms of the extent to which you think they give students an opportunity to communicate a deeper understanding of mathematics.

Ms. Barker's Item:

Ellen says that it is impossible for the composition of two reflections not to be a translation. Mike says that she is wrong. Who is right and why?

Mr. Walker's Item:

Find the length of the longest umbrella that can fit inside of a suitcase whose interior dimensions are 6" by 24" by 30".

Ms. Burn's Item:

For what value(s) of *b* will the graph of $f(x) = x^2 \cdot bx - 12$ be tangent to the *x*-axis?

Mr. Klein's Item:

Ms. Jackson is having a concrete driveway put into her new home. The original plan calls for the driveway to be 8 feet wide and 50 feet long with a depth of 3 inches. If she changes the dimensions to 10 feet wide and 60 feet long (same depth), how much more should the bigger driveway cost compared to the original estimate?

Mr. Murphy's Item:

Mr. Allen is going to invest $1,000 in either stocks or a certificate of deposit. What questions and what information should Mr. Allen obtain in order to make an informed decision?

As you read other portions of *Mathematics, Pedagogy, and Secondary Teacher Education*, notice that the authors make use of assessment in many nontraditional ways. It is rare that you will be asked to engage in an activity with a right or wrong answer. Frequently, the authors suggest ways of reexamining and consolidating what you have learned via strategies other than those associated with paper and pencil tests. For example, you will find activities involving group work, interviews with your classmates, essay writing, problems, and reflective situations—all of which contain important elements of assessment built in.

WHAT DOES IT MEAN TO BE A "PROFESSIONAL TEACHER OF MATHEMATICS"?

As either a preservice teacher or an experienced teacher, you are at a particular stage of development in your professional career. You might wonder what it means for you to be a professional in the first place. After all, we don't refer to everyone as a professional. While many definitions exist, we generally think of professional teachers as those who have a certain knowledge base about teaching and learning that sets them apart from those engaged in other aspects of education. We think it is important for teachers to develop a knowledge base that makes them the experts in revising and shaping curricula for their students. Some call this *empowerment* in that teachers have the ability and the responsibility to create learning environments that encourage students to see connections within mathematics and between mathematics and the real world. This ability has to do with seeing the connectedness and the structure of mathematics and how that structure can be related to activities for students. It has to do with the means by which we engage students in the act of reflecting on their own learning and how they can develop self-generative activities so that they can explore mathematical ideas on their own. It has to do with understanding psychological principles that underlie the teaching and learning of mathematics. Foremost, we believe that it has to do with the notion of integrating content and pedagogy so that students learn mathematics in a way that is consistent with the processes of reasoning, communication, and problem solving.

More than fifteen years ago Fletcher (1979) raised the question, "Is the teacher of mathematics a mathematician or not?" His response was affirmative, using the following reasoning process. Mathematicians have a generalized knowledge of mathematics and a specialized knowledge in a particular area of mathematics. Similarly, Fletcher argues, teachers of mathematics should have a generalized knowledge of mathematics and a specialized knowledge that allows them to adapt and modify existing curricula to better accommodate their students' learning of mathematics. The very purpose of *Mathematics, Pedagogy, and Secondary Teacher Education*, is to help teachers develop this kind of knowledge so that they can become flexible and adaptive teachers in their classrooms.

As you read through these pieces, see if you can identify qualities that you associate with being a professional mathematics teacher. In "Developing a Topic Across the Curriculum," you will encounter Ms. Lopez. Not only does she provide a context for students to see mathematical functions as a vehicle for modeling real-world phenomena, but she has also acquired ways of listening to students and of encouraging them to learn from each other. Her colleague, Mr. Washington, maintains that the primary purpose of teaching mathematics is to enable students to see the logical structure of mathematics. In what ways is his professional orientation different from Ms. Lopez's? Other teachers you will meet include Mr. Kubiack, who teaches his students explicit techniques for using functions to model real-world phenomena, and Mr. Black and Ms. Waters, who not only share a common vision of teaching mathematics but also work together in a variety of contexts, including the development of a proposal to the National Science Foundation. Consider how these teachers might have responded to the questions posed in the sections above and how their responses might be similar to or different from yours. You will be introduced to students such as Dirk and Stefan, who are interviewed about their understanding of the Pythagorean theorem, and Carola and Andrea, who

are engaged in the process of solving intriguing counting problems. If these students were in your classes, how would you assess their understanding of mathematics? Stories and anecdotes about these teachers and students are offered to invite you to define what mathematics means to you and how you can use your knowledge about mathematics and pedagogy in an integrated way. We offer them to assist you in your professional development.

Return now to Reflective Problem 3. Consider your responses and complete the following activity.

Reflective Problem 5

Revisit your response for Reflective Problem 3. Do you think your earlier response still captures your view of mathematics? Why or why not?

We conclude with the following reflective problem.

Reflective Problem 6

Thumb back through the chapter and identify those ideas that you think represent the biggest challenge you will face in the teaching of mathematics.

Write a two- to three-page paper on why you see these ideas as challenging as you think about your role as a teacher of mathematics.

You may want to save the paper and revisit it at a later time in your teaching career.

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