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Small-Group Cooperative Learning in Mathematics

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SMALL -group cooperative learning can be used to foster effective mathematical communication, problem solving, logical reasoning, and the making of mathematical connections—all key elements of NCTM's *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). Small-group cooperative learning methods can be applied with all age levels of students, all levels of the mathematics curriculum from elementary school through graduate school, and all major topic areas in mathematics. Moreover, small groups working cooperatively can be used for many different instructional purposes: in the discussion of concepts, inquiry/discovery (often using manipulative materials), problem solving, problem posing, proofs of theorems, mathematical modeling, the practice of skills, review, brainstorming, sharing data from different groups, and the use of technology.

This article will address many of the questions most frequently asked by teachers about cooperative learning: (1) What is the rationale for small-group cooperative learning in mathematics? (2) What are the outcomes of cooperative small-group learning in mathematics? (3) What are appropriate leadership styles for the teacher? (4) How does one foster cooperative behavior among students? (5) How are groups formed? (6) How are students held accountable and graded? (7) How frequently do group activities occur? (8) What types of mathematical activities are most appropriate for small-group learning? (9) What resource materials for group work are available to the teacher? (10) What do students and teachers perceive as the strengths and limitations of cooperative learning in mathematics? (11) How does a teacher begin group work in mathematics?

RATIONALE

Why does cooperative learning have a place in mathematics instruction? The learning of mathematics is often viewed as an isolated, individualistic, or competitive matter—one sits alone and struggles to understand the material or solve the assigned problems. This process can often be lonely and frustrating. Perhaps it is not surprising that many students and adults are afraid of mathematics and develop "math avoidance" or "math anxiety." They often believe that only a few talented individuals can function successfully in the mathematical realm.

Small-group cooperative learning addresses these problems in several ways:

- Small groups provide a social support mechanism for the learning of mathematics. "Small groups provide a forum in which students ask questions, discuss ideas, make mistakes, learn to listen to others' ideas, offer constructive criticism, and summarize their discoveries in writing" (NCTM 1989, p. 79).
- Small-group learning offers opportunities for success for all students in mathematics (and in general). The group interaction is designed to help all members learn the concepts and problem-solving strategies.
- Mathematics problems are ideally suited for group discussion because they have solutions that can be objectively demonstrated. Students can persuade one another by the logic of their arguments.
- Mathematics problems can often be solved by several different approaches, and students in groups can discuss the merits of different proposed solutions.
- Students in groups can help one another master basic facts and necessary computational procedures in the context of games, puzzles, or the discussion of meaningful problems.
- The field of mathematics is filled with exciting and challenging ideas that merit discussion. One learns by talking, listening, explaining, and thinking with others, as well as by oneself.
- Mathematics offers many opportunities for creative thinking, for exploring open-ended situations, for making conjectures and testing them with data, for posing intriguing problems, and for solving nonroutine problems. Students in groups can often handle challenging situations that are well beyond the capabilities of individuals at that developmental stage.

OUTCOMES

The outcomes of cooperative learning methods have generally been quite favorable. General reviews of research have been presented by Sharan, Slavin, and the Johnsons; these are described in Slavin et al. (1985). Reviews by Davidson (1985), Davidson and Dees (forthcoming), and Webb (1985) focus specifically on mathematics learning; these first two mathematics reviews address achievement and the third addresses group interaction.

Research has shown positive effects of cooperative learning in the following areas: academic achievement, self-esteem or self-confidence as a learner, intergroup relations including cross-race friendships, social acceptance of mainstreamed children, and ability to use social skills (if these are taught).

Davidson (1985) and Davidson and Dees (forthcoming) reviewed about eighty studies in mathematics comparing student achievement in cooperative learning versus whole-class traditional instruction. In over 40 percent of these studies, students in the small-group approaches significantly outscored the control students on individual mathematical performance measures. In only two studies did the control students perform better, and both of these studies had irregularities in design.

The effects of cooperative learning of mathematical skills were consistently positive when there was a combination of individual accountability and team recognition for commendable achievement. The effects of small-group learning were nonnegative (i.e., not significantly different from traditional instruction) if the teacher had no prior experience in small-group learning, was not aware of well-established methods, and did very little to foster group cooperation or interdependence.

For many teachers the social benefits of cooperative learning are at least as important as the academic effects. Cooperative learning is a powerful tool for increasing self-confidence as a learner and for fostering true integration among diverse student populations.

CLASSROOM PROCEDURES

Cooperative learning involves more than just putting students together in small groups and giving them a task. It also involves giving very careful thought and attention to various aspects of the group process.

A class period might begin with a meeting of the entire class for an overall perspective. This meeting may include the presentation of new material, class discussion, posing problems or questions for investigation, and clarifying directions for the group activities.

The class is then divided into small groups, usually with four members apiece. Each group has its own working space, which might include a flip chart or a section of the chalkboard. Students work together cooperatively in each group to discuss mathematical ideas, solve problems, look for patterns and relationships in sets of data, make and test conjectures, and so on. Students actively exchange ideas and help each other learn the material. The teacher takes an active role, circulating from group to group, giving assistance and encouragement, and asking thought-provoking questions as needed.

The Teacher's Role

In each type of small-group learning, a number of leadership and management functions must be performed. Although some of them may be explicitly delegated to the students, these functions are generally handled by the teacher and include the following:

- Initiating group work
- Presenting guidelines for small-group operation
- Fostering group norms of cooperation and mutual helpfulness
- Forming groups
- Preparing and introducing new material
- Interacting with small groups
- Tying ideas together
- Making assignments of homework or in-class work
- Evaluating student performance

Each of these functions can be performed in different ways and to varying degrees, depending on the model of small-group instruction in effect.

The behavior of the teacher will vary for different phases of instruction, such as introducing new material, facilitating group activities, and summarizing. A teacher with small groups can introduce new material and pose problems and questions for discussion or investigation (*a*) orally, with a whole-class discussion at the beginning of a period or with individual groups at appropriate moments; or (*b*) in written form, with teacher-made worksheets or special texts designed for small-group learning. In any event it is essential to make sure that students understand the mathematical problem situation. (Several examples of activities are given later.)

The teacher provides guidance and support during small-group activities, observing the group interaction and their solutions on the board, and, while visiting particular groups, checking their solutions, giving hints, clarifying notation, making corrections, answering some questions, and so on. The teacher also performs social functions, such as providing encouragement, drawing members into the discussion, and helping the groups function more cooperatively. The teacher should behave in a friendly and constructive manner and strike a balance between giving too much and too little assistance. It may take a while for the teacher to become adept at observing, diagnosing difficulties, and intervening in a facilitative way.

In summarizing discussions with the whole class, the teacher may need to answer certain questions, serve as discussion moderator, and clarify and summarize what the students have found. An overall synthesis by the teacher is needed from time to time, since students in the groups sometimes "see the trees but lose sight of the forest." Occasionally it is useful to hear brief summary statements from different small groups. This is especially powerful when different groups have tackled different aspects of a complex situation, perhaps leading to a generalization. However, end-of-period summaries are not always necessary and should not become a ritual.

Fostering Cooperation

To help the students learn how to cooperate with one another, the teacher might present a set of guidelines for group behavior:

1. Work together in groups of four.
2. *Cooperate* with other group members.
3. Achieve a group solution for each problem
4. Make sure that everyone understands the solution before the group goes on.
5. Listen carefully to others and try, whenever possible, to build on their ideas.
6. Share the leadership of the group.
7. Make sure that everyone participates and no one dominates.
8. Take turns writing problem solutions on the board.
9. Proceed at a pace that is comfortable for your own group.

Different sets of guidelines are possible. Difficulties in group interaction can usually be analyzed and cleared up by reminding students about the guidelines, whichever ones are stated.

One way to improve group functioning is to reflect on the communication and interaction process occurring in each group. Each member of the group can address the following questions while other members listen: How are you contributing to the successful operation of this group? What can you do to make it function even better?

A more elaborate set of questions for discussion follows: (1) Did your group achieve at least one solution to the problem or task? (2) Did everybody understand the solution? (3) Did people ask questions when they didn't understand? (4) Did people give clear explanations? (5) Did everyone have a chance to contribute ideas? (6) Did people listen to one another? (7) Did any one person take over the group? (8) Did the group really work together on the task? (9) Was there enough time for exploration?

Group Size and Formation

It is necessary to form work groups of small size, since the opportunity for active participation decreases as the group size increases. In mathematics classes, groups with four members seem to work best. They are large enough to generate ideas for the discussion and solution of challenging problems, yet not be decimated by the absence of one member. They are small enough to permit active participation, to allow clustering around a chalkboard panel, and not to require a leader or elaborate organizational

structure. Groups of four can also split into pairs for occasional computational practice or simple application problems.

Generally, if one gives great care and attention to forming the groups, the groups will function better and there will be less need to switch groups frequently. Although groups can be formed by random assignment, heterogeneous grouping ensures a mixture of mathematical achievement, gender, and race/ethnicity. An occasional use of sociometric choice by students is possible, but homogeneous grouping is usually not recommended. Proponents of different schemes sometimes express strong ideological viewpoints about grouping procedures.

An experienced teacher can usually work comfortably with as many as six or seven groups but might feel overly extended with eight groups. In very large classes, the teacher may need an aide for help with group supervision; a more advanced student can often be an effective aide.

Evaluation

A variety of grading schemes are compatible with small-group instruction, including in-class tests and quizzes, take-home tests, homework, classwork (consider attendance, participation, and cooperation), self-evaluation, and peer evaluation. If a teacher gives tests on a specific date, that date should realistically allow all groups to have finished the material before the test date without rushing. If teachers give grades for classwork (including attendance, participation, cooperation) they should *not* grade individual mathematical performance during class; doing so will foster competition and destroy cooperation. Some teachers have found that evaluation measures such as group projects or the occasional use of group tests on which all members receive the same grade work well. Personal philosophy has a great bearing on these decisions.

Frequency of Use

The small-group method can be used as a total instructional system or in combination with other methods. Groups can be used all the time, on specific days of the week, during portions of any class period, or for specific topics. I personally prefer to use small groups for most of the class time, except in a few multisection departmentalized courses taught on a rigid time schedule at breakneck pace with uniform hour exams.

MATHEMATICAL ACTIVITIES

Let us now focus on curriculum-related issues. What constitutes a good problem for small-group discussion? Crabill (1990) suggests that a good problem for group exploration has the following attributes (pp. 215–16):

> (1) It is presented in a meaningful context and in everyday language whenever possible . . . . (2) It is easy to state. That is, the problem is clearly defined, even though it may not be easy to solve. (3) It is easy to visualize physically. That is, it is not abstract, even though it may lead to an abstract generalization later. (4) It stimulates student questions that may be better than the original question. This is the most important attribute. These attributes of a good problem are especially valuable when starting a new topic or starting new learning groups.

Two examples of effective problems for small groups follow.

Activity 1

Students in small groups are capable of "discovering" several classical summation formulas. It is best to start with a simple situation, such as the following:

Consider sums of consecutive odd integers, such as 1, 1 + 3, 1 + 3 + 5, 1 + 3 + 5 + 7, 1 + 3 + 5 + 7 + 9, ... Compute these sums and record the answers in this table:

Number of terms	1	2	3	4	5	6	7
Sum of odd integers	1	4					

State a formula (rule, generalization) that tells how to compute easily the sum of any number of consecutive odd integers (starting with 1). Use your rule to compute the sum of the first 1000 odd integers.

Now let's look for other formulas for the sums of consecutive—

- even integers: 2, 2 + 4, 2 + 4 + 6, 2 + 4 + 6 + 8, . . .
- integers: 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, . . .
- integers cubed: 1^3, $1^3 + 2^3$, $1^3 + 2^3 + 3^3$, $1^3 + 2^3 + 3^3 + 4^3$, . . .

Compute these sums and record the answers in the following table:

Number of terms	1	2	3	4	5	6	7
Sum of even integers	2	6					
Sum of integers	1	3					
Sum of cubes	1	9					

Hint (if needed): Compare different lines in the table.

State three formulas (rules, generalizations) that tell how to compute easily (1) the sum of any number of consecutive even integers (starting with 2), (2) the sum of any number of consecutive integers (starting with 1), and (3) the sum of the cubes of any number of integers (starting with 1^3). Use your rules to compute each of these sums, with each having 1000 terms.

Comment: If students are familiar with the use of variables, they may write their formulas in symbolic notation:

$$1 + 3 + \cdots + (2n - 1) = n^2$$
$$2 + 4 + \cdots + 2n = n(n + 1)$$
$$1 + 2 + \cdots + n = n(n + 1)/2$$
$$1^3 + 2^3 + \cdots + n^3 = [n(n + 1)/2]^2$$

Otherwise, they may simply state their results in words.

Activity 2

Sometimes a situation can be divided into tasks of roughly equal difficulty. Each group explores one such task or aspect of the situation and then presents its results to the whole class. For example, in first introducing the concept of the slope of a line, the teacher can ask each group to make several graphs on the same axis by plotting points. The assignments can be as follows:

Group 1: Graph $y = x, y = 2x, y = 3x$
Group 2: Graph $y = -x, y = -2x, y = -3x$
Group 3: Graph $y = (1/2)x, y = (1/3)x, y = (1/4)x$
Group 4: Graph $y = (-1/2)x, y = (-1/3)x, y = (-1/4)x$

Each group then displays its graphs on the chalkboard or on large pieces of paper. These displays allow students to see the effects of changing the coefficient of x and permit generalizations regarding these effects on the slope of the line. Similar explorations can be conducted with intercepts.

One of the major issues in designing group activities is the amount of guidance (or structure) to give in the directions. Higher levels of guidance imply lower levels of open-ended exploration, inquiry, or discovery and perhaps less intellectual excitement and "thrill of discovery," but they also provide for more efficient use of time, more rapid coverage of material, and less student frustration. There are trade-offs in any decision about levels of guidance. Other variables to be considered are the amount of time available, the perceived ability of the students to handle challenges and possible frustration, and the importance of open-ended exploration for the particular topic at hand. It is often hard to gauge this in advance; classroom trials may be needed to redesign the activities according to the actual experiences of the groups. See Davidson, McKeen, and Eisenberg (1973) for further suggestions. A number of people have created mathematics curriculum resources for use in cooperative learning situations at different levels; see Davidson (1990) for detailed descriptions.

Perceptions of Teachers and Students

In attitude surveys given over a period of years, the main problems that teachers and students expressed about cooperative learning in mathematics were as follows: Concerns about covering enough material, initial difficulties in forming effective groups, barriers to fostering cooperation among students, occasional conflict or frustration with overly difficult mathematical problems, providing high-quality instructional materials, and handling a major shift in the roles of teacher and student. Although student attitudes toward this method of instruction are generally favorable, the degree to which they are favorable depends on the teacher's experience and skill in handling the problem areas mentioned above.

There are many advantages to learning mathematics in small, cooperative groups. The following positive points are frequently mentioned by teachers and students in responding to attitude surveys: Students are actively involved in learning mathematics while working at a comfortable pace. They learn to cooperate with others, to improve their social skills, and to communicate in the language of mathematics. The classroom atmosphere tends to be relaxed and informal, help is readily available, questions are freely asked and answered, and misconceptions become quickly apparent and are readily resolved. Students tend to become friends with their group members across traditional boundaries of race, ethnicity, or sex. The teacher-student relationship tends to be more relaxed, pleasant, and closer than in a traditional approach. Teachers benefit from some intellectual companionship with their students and often find themselves invigorated professionally and less prone to burnout. The usual disciplinary problems of talking and moving around are eliminated by definition. In addition, many students maintain a high level of interest in the mathematical activities. Students are not bored in class; many of them like mathematics more than when involved in teacher-centered approaches. Finally, students have an opportunity to pursue the more challenging and creative aspects of mathematics and to become more confident problem solvers while acquiring at least as much information and skill as when they are taught with more traditional approaches.

GETTING STARTED

Here are a few tips for teachers just getting started in implementing cooperative learning: (1) Begin with a simple approach such as think-pair-share. The teacher poses questions to the class, where students are sitting in pairs. Students *think* of a response individually for a given period of time,

then *pair* with their partners to discuss the question and reach consensus. The teacher then asks students to *share* their agreed-on answers with the rest of the class. This method can be used during teacher presentations whenever students appear to be confused or not attentive. Students are asked to discuss the content and attempt to explain it to one another or else formulate a question for the class. The method can also be used for previously planned brief discussions or short practice activities. It has a powerful, immediate effect in livening up the class. (2) Look for opportunities within the regular curriculum to use groups. For example, class time that is normally devoted to individual seatwork can be changed into group work. The textbook may include a good selection of exercises and problems for this purpose. (3) Give very clear, step-by-step directions and check to make sure that students understand them. (4) Start with a class that you think will respond favorably to cooperative learning. (5) Don't feel that you must establish very tight control for weeks before beginning group activities. Group work can become part of your management system. (6) Clearly inform the principal or building administrator and parents what you are doing and why you are doing it. (7) If at all possible, find a colleague who will use similar methods. Two teachers together can provide strong mutual support. (8) Expect that group activities will not necessarily go smoothly at first; it usually takes two or three weeks for students to begin functioning well in groups. (9) Remember that change is a gradual process, not an event. Don't try to change everything all at once.

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