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The Four Faces of Mathematics

Keith Devlin

MATHEMATICS presents four faces to the world:

1. Mathematics as computation, formal reasoning, and problem solving
2. Mathematics as a way of knowing
3. Mathematics as a creative medium
4. Applications of mathematics

For the most part, current grades K–12 education concentrates on the first face and makes some reference to the fourth face but pays little or no attention to the remaining two faces. In this article, I shall argue that as we enter the twenty-first century, mathematics education should display all four faces.

WHY EDUCATE?

What is the purpose of a school education? In times past, it might have been possible to answer that at least one function was to provide young people with (at least some of) the knowledge and skills required for different occupations. Even if such a view was once defensible, it surely does not apply today, when most occupations require very specialized knowledge and skills and where the pace of change is such that no one can predict more than a few years in advance what the most important knowledge and skills are going to be—save for the crucial importance of one particular ability: the ability to adapt to changing circumstances and to acquire rapidly a new set of skills.

A much better answer, in my view, is that one purpose of a school education is to develop in young people *the ability to acquire specialized knowledge and skills* as and when they need them during the course of their working lives.

I think the second answer is much better. But I also think that preparation for work is just one part of the real purpose of a school education. That real purpose, as I see it, is to pass on the main elements of our culture and to pre-

pare young people to lead full and active lives, playing their full roles as citizens. This goal is very broad and ambitious, probably not one that a poor or developing nation can afford. But it is, I suggest, the only goal appropriate for any society, such as ours, that has become sufficiently affluent for most people to have considerable choice about how they want to live their lives.

The question I want to investigate in this essay is, What are the implications of this view of education when it comes to mathematics?

WHY MATHEMATICS?

Why does mathematics form a part of the education we provide for our young? Let me start my answer, as I did above for education in general, by concentrating on the utilitarian aspect of mathematics: How much mathematics does a typical young person *need* to learn to be adequately prepared for his or her subsequent occupations?

The simple answer is, much less than is popularly assumed. Few citizens in modern society ever need or make real use of any appreciable knowledge of, or skill in, mathematics. What mathematics they do need and use they have already met by the time they are twelve years old. (It is interesting that prior to that age, the majority of children declare that they "like" mathematics.)

The continuation of modern society, however, requires a steady supply of a small number of individuals having considerable training in mathematics. Just as in the industrial age we burned fossil fuels to drive the engines of production, so in today's information age the principal fuel we burn is mathematics. Whatever else we do with mathematics education, in order that the future supply of mathematicians does not dry up, we must ensure that all high school and university students are made aware of the nature and importance of mathematics so that those who find they have an interest in the subject, and an aptitude for it, can choose to study it in depth.

The two observations above have an obvious consequence for grades K–12 mathematics education: a major goal should be to create an awareness of the nature of mathematics and the role it plays in contemporary society. Grades K–12 mathematics should therefore be taught much more like history or geography or English literature—not as a utilitarian toolbox but as a part of human culture. Remember that the goal I am advocating is to produce an educated citizen, not a poor imitation of a twenty-dollar calculator. An educated citizen should be able to answer the following two questions about mathematics:

- What is mathematics?
- Where and how is mathematics used?

Few people can answer either question completely. In a moment I shall give my own answers to these two questions—answers that I believe an adequate mathematical education should equip everyone to provide. In the meantime, let me observe that a shift away from a largely procedure-oriented approach to mathematics to one that encompasses mathematics as a part of human culture—as a broad body of human knowledge and a "way of knowing"—is very much in keeping with the overall philosophy of the purpose of school education I stated at the beginning of the article. And yet now I have reached the same conclusion from the more obviously utilitarian standpoint of preparation for future careers.

In order to give our students a broad view of mathematics as a part of human culture that we need to address all four faces of mathematics. How do we go about doing so?

The Familiar Face

The first face of mathematics (computation, formal reasoning, and problem solving) is the one most familiar to the majority of people. This is the face we generally concentrate on when we think of mathematics education. Although it is so familiar that I will not dwell on it in this article, I am in no way advocating that we abandon this aspect of mathematics. Although the pocket calculator and the computer have made it unnecessary to spend large amounts of time developing skill in mental arithmetic, basic arithmetic and a good sense of number are surely invaluable skills for everyone to have in today's world. Moreover, learning mathematics seems to provide a general benefit in developing the mind that cannot be obtained any other way. For instance, even though few people ever make explicit use of high school algebra, it has been shown (see, e.g., U.S. Department of Education [1997]) that not only do students who complete a rigorous high school course in algebra or geometry more frequently gain entry into a college or university, but they perform much better once they are there whatever they choose to study. In short, completing a rigorous course in mathematics—it is not even necessary that the student do well in such a course—appears to be an excellent means of sharpening the mind and developing skills in problem solving and analytic thinking.

Despite the benefits, however, if we continue to concentrate largely on the first face of mathematics and ignore the remaining three faces, we not only fail our students—our future citizens—we also do a great disservice to one of humankind's most towering cultural achievements.

Mathematics as a Way of Knowing

Thought of as a human activity, mathematics is a particular way of knowing, a way of understanding different aspects of the world we live in. Mathematics is not the only way of knowing by any means: biology, chemistry, physics, psychology, sociology, linguistics, poetry, painting, sculpting, playwriting, and novel writing are just some of a great many other ways of knowing. Mathematics, however, is a way of knowing that has proved to be inordinately successful. Today, scarcely any aspect of our lives is not affected, often in a fundamental and far-reaching way, by the products of mathematics. When you think of the technological and communications infrastructure that undergirds our lives, you realize that we are in fact living in a "mathematical universe."

Each of the different ways of knowing has its own particular characteristics. What exactly is the way of knowing we call *mathematics?* Most people think that mathematics is merely a collection of rules for manipulating numbers, but they are wrong. The manipulation of numbers is just one very small part of mathematics. The simplest, most accurate description of mathematics I know is this: Mathematics is the science of patterns. The mathematician looks at a certain aspect of the world and strips away the complexity, leaving an underlying skeleton. Looking at different aspects of the world in this way leads to different branches of mathematics, which focus on different kinds of patterns. For instance:

- Arithmetic and number theory study the patterns of number and counting.
- Geometry studies the patterns of shape.
- Calculus allows us to handle patterns of motion.
- Logic studies patterns of reasoning.
- Probability theory deals with patterns of chance.
- Topology studies patterns of closeness and position.
- Projective geometry arises from a study of the patterns that enable us to perceive depth in a two-dimensional picture.
- Group theory results from a study of the patterns of symmetry.

It would of course be impractical to try to teach students how to do each of these different kinds of mathematics (to say nothing of all the branches I have not listed), and I am not advocating doing so. Rather, I think it is important to make our students aware of the existence of these many different branches of the subject and of what makes them all *mathematics*.

How do we go about creating that awareness? My answer is that we do so by describing some of the many applications of mathematics. In so doing, we show our students that mathematics works by *making the invisible visible*. We show them that by giving us a means to "see" (and hence to understand) things that would otherwise be invisible, mathematics demonstrates that it is one of the most amazing constructions of the human mind, a powerful testimony to human ingenuity and intellectual creativity. The following are just a few examples of the kind of thing I have in mind.

Without mathematics, there is no way to understand what keeps a jumbo jet in the air. As we all know, large metal objects don't stay above the ground without support. But we can't see anything holding up a jet aircraft. It takes mathematics to help us "see" what keeps an airplane aloft. In this instance, what lets us "see" the invisible is an equation discovered by the mathematician Daniel Bernoulli early in the eighteenth century.

What causes objects other than aircraft to fall to the ground when we release them? "Gravity," you answer. But you are simply giving the phenomenon a name. We might as well call it magic. It's still invisible. To understand it, we have to "see" it. That's exactly what Newton did with his equations of motion and mechanics in the seventeenth century. Newton's mathematics enabled us to "see" the invisible forces that keep the earth rotating around the sun and cause an apple to fall from a tree onto the ground.

Both Bernoulli's equation and Newton's equations use calculus. Calculus works by making visible the infinitesimally small. That's another example of making the invisible visible. Here's another:

Two thousand years before we could send spacecraft into outer space to take pictures of our planet, the Greek mathematician Eratosthenes used mathematics to show that the earth was round. Indeed, he calculated its diameter, and hence its curvature, with 99 percent accuracy. Today, we may be close to extending Eratosthenes' feat and discovering whether the universe is curved. Using mathematics and powerful telescopes, we can "see" into the outer reaches of the universe. According to some astronomers, we will soon see far enough to be able to detect and measure any curvature in space.

If we can calculate the curvature of space, then we can use mathematics to see into the future to the day the universe comes to an end. Using mathematics in conjunction with scientific theories, we have already been able to see into the distant past, making visible the otherwise invisible moments when the universe was first created in what is called the *big bang*.

Back on earth, what makes the pictures and sound of a football game miraculously appear on a television screen on the other side of town? One answer is that the pictures and sound are transmitted by radio waves—a special case of what we call *electromagnetic radiation*. But as with gravity, the term just gives the phenomenon a name; it doesn't help us "see" it. In order to "see" radio waves, we have to use mathematics. Maxwell's equations, discovered in the last century, make visible to us the otherwise invisible radio waves.

Mathematics has been used to describe human patterns, as well:

- Aristotle used mathematics to try to "see" the invisible patterns of sound that we recognize as music.
- Aristotle also used mathematics to try to describe the invisible structure of a dramatic performance.
- In the 1950s, the linguist Noam Chomsky used mathematics to "see" and describe the invisible, abstract patterns of words that we recognize as a grammatical sentence. He thereby turned linguistics from a fairly obscure branch of anthropology into a thriving mathematical science.

Finally, using mathematics, we are able to look into the future:

- Probability theory and mathematical statistics help us predict the outcomes of elections, often with remarkable accuracy.
- We use calculus to predict tomorrow's weather.
- Market analysts use various mathematical theories to try to predict the future behavior of the stock market.
- Insurance companies use statistics and probability theory to predict the likelihood of an accident during the coming year, and they set their premiums accordingly.

Mathematics allows us to make visible another invisible when it helps us predict the future. In that situation our mathematical vision is not perfect; our predictions are sometimes wrong. But without mathematics, we cannot see even poorly.

Let me stress once again that I am not advocating that we teach our students how to perform the mathematics involved in the applications mentioned above. Rather, along with the mathematics that we require our students to carry out, we should also *describe* some of the many other (perhaps more advanced) branches of the subject and give examples of the different ways they can be applied. Just as it is not necessary to know how to build or repair a car in order to take a tour in the country, so too it is not necessary to know how to do mathematics in order to understand how it is used. This analogy works just as well for the third face of mathematics, to which I turn next.

Mathematics as a Creative Medium

Few people are aware of the breadth of modern mathematics. Even fewer people realize that mathematics can also be used as a creative medium, in much the same way that a sculptor uses stone, a painter uses paint and canvas, or a novelist uses language. Used as a creative medium, mathematics again makes the invisible visible. In this instance, it takes the creative ideas produced in our minds, which are invisible to others, and makes them accessible to public perception, so others can share in them and experience our ideas. Mathematics as a creative medium is the third face of mathematics.

Arguably the first major creative use of mathematics occurred in the Renaissance, when artists discovered how to show depth in a two-dimensional painting. Artists refer to the trick as *the rules of perspective;* mathematicians call it *projective geometry*. Whatever the terminology, the underlying idea is to discover and use a "geometry"—the geometry of vision.

Similar to the artists of the Renaissance, present day artists have learned to use a geometry of light (so-called *ray tracing*) to produce realistic-looking computer graphics for the movie industry, images with surface texture, highlighting, and shadow.

More generally, much of the digital special-effects work in today's movie industry results from the use of mathematics as a creative medium. The special-effects artists use computers to create and manipulate mathematical descriptions of images that become visible only at the end of the process, when the massive arrays of numbers generated and stored in the computer are turned into colored pixels on a screen or a film.

Long before we had computers, the British writer Edwin A. Abbott, in his delightful novella *Flatland* (Abbott 1991), used the geometry of two and three dimensions as the vehicle for an insightful satirical commentary on the social mores of Victorian England. In the present era, the playwright Tom Stoppard has used mathematics as a vehicle for commenting on society in a number of his plays, of which *Arcadia* (Stoppard 1996) is a prime example.

Both Abbott and Stoppard provide examples of creative uses of mathematics not by mathematicians but by artists—artists who might well declare themselves ignorant of mathematics (although in most instances their ignorance is not so much of mathematics but of the fact that what they are doing has mathematical aspects). In another example, the discovery of non-Euclidean geometries and the investigations of four-dimensional geometry in the nineteenth century inspired many artists to explore and experiment with the nature of space and of dimension. A notable artistic development of this kind, with quite obvious mathematical roots, was the cubist movement in painting, led by Pablo Picasso and others. The Dutch artist M. C. Escher was another who tried to express different geometries in his paintings and etchings. Escher did in fact study mathematics, and he sometimes made the mathematics in his work fairly explicit.

More recently, the artist Tony Robbin (see Devlin [1998]) has spent a large part of his career trying to depict four-dimensional space on a two-dimensional canvas—a sort of "super perspective," if you will. According to Robbin, one of the main functions of art is to reflect on, comment on, and thereby help us understand various aspects of life. He sees his own work in exploring higher-dimensional spaces as a way to visualize and understand the multidimensional complexities of life in multiracial and multicultural societies.

Virtual reality art is another domain of modern art that is heavily dependent on mathematics. Artists such as Marcus Novak (see Devlin [1998]) use mathematics to create "immersive experiences" in which the user dons a stereoscopic and stereophonic helmet and wears a special pointing glove to "step inside and explore" the artistic world that the artist has created using mathematics, a world that can be multidimensional or structured according to a geometry very different from the one we are familiar with from our everyday world.

Of course, the use of mathematics as a creative medium is an application of mathematics (and so too is its use as a way of knowing). So we have already encountered the fourth face of mathematics: applications of mathematics. But I want to consider the application of mathematics separately, as a face in its own right.

Applications

Mathematics education must include applications. In order to live a full life, everyone needs to have an awareness of what goes into making that life possible. In particular, modern society depends on the many applications of mathematics that have been developed over the centuries, especially over the past half century. When we travel by car, train, or airplane, we enter a world that depends on mathematics. When we converse on the telephone, or attend a major sporting event, we are enjoying the products of mathematics. When we listen to music on a compact disc or log on to the Internet, we are using the products of mathematics. When we go into the hospital or take out insurance, we are depending on mathematics. As educators, we owe it to our students to make them aware of the scope, the depth, and the profound impact of the applications of mathematics in today's world.

Of course, much of the mathematics that lies behind our everyday world is advanced and highly specialized, and there is certainly no need for all but a small number of experts to understand that mathematics. Consequently, most classroom discussions of the applications of mathematics will have to be just that—discussions. But not all applications of mathematics are inaccessible. Moreover, with improving computer technology, it is getting easier for us to have our students actually carry out some well-chosen applications of mathematics. Thus, we are not restricted solely to talking about applications; we can have our students carry out some applications—an activity that should definitely be included along with the "tour" of the other applications I am advocating.

Appreciating Mathematics

By changing our mathematics education system radically so that the primary goal for the vast majority of students is to create an awareness of the what, the how, and the why of mathematics rather than develop skills that only a tiny minority of the students will ever use, we will achieve two important goals:

1. That tomorrow's citizens appreciate the pervasive role played by one of the main formative influences on the culture in which they live.
2. That the individuals who turn out to have an interest in, and a talent for, advanced mathematics be exposed to the true nature and extent of the subject at an early age and as a result have an opportunity to pursue their interest to the eventual benefit of both themselves and society as a whole.

The justification for goal 1 is simply this: A human being is the richer for having a greater understanding of the nature of her or his life. The more ways we have to know our world and ourselves, the richer our lives are.

Regarding goal 2, success at high school mathematics, including calculus, is not always a good predictor of later success in mathematics. Mathematics through calculus is largely (though not exclusively) algorithmic: A successful strategy adopted by many students is simply to learn various rules and procedures and know when and how to apply them. In contrast, much (though not all) college-level mathematics beyond calculus is highly creative, requiring original thought and the ability to see things in novel ways. Since the creative mathematician often needs to apply rules and use algorithmic thinking, many successful mathematicians have indeed excelled in their high school mathematics classes. But many students who shone at high school mathematics find that they struggle with, and eventually give up, the subject in college, when they discover that algorithmic ability alone is not enough. And the fact that some of the very best professional mathematicians did poorly at high school mathematics but by some fluke were drawn to the discipline later in life suggests that our present system of school mathematics education probably turns off a significant number of students who have the talent for later mathematical greatness.

QUANTITATIVE LITERACY

The focus of this article has been on mathematics education, primarily for the vast majority who do not make explicit use of mathematics in their lives. I have largely ignored what is often referred to as *quantitative literacy*. It is important to recognize that mathematics and quantitative literacy are not the same. Roughly speaking, quantitative literacy—sometimes called *numeracy*—comprises a reasonable sense of number, including the ability to estimate orders of magnitude within a certain range, the ability to understand numerical data, the ability to read a chart or a graph, and the ability to follow an argument based on numerical or statistical evidence. (See, e.g., Steen [1997].)

Since people often confuse quantitative literacy with mathematics, I shall address the former, even though it is not the focus of this article.

In today's society, numeracy is a fundamental life skill, on a par with literacy. In consequence, just as literacy is every teacher's responsibility—to be developed at all times in every lesson, not just in the English class—so too is numeracy every teacher's responsibility. It is as much the responsibility of the teacher of history, of home economics, or of physical education as it is the responsibility of the teacher of mathematics or science. Regarding basic quantitative skills as somehow separate from basic language skills sends quite the wrong message to our students. Confusing quantitative literacy with mathematics simply confounds the problem.

According to some estimates, fewer than 10 percent of the adult population of the United States is quantitatively literate. Such figures are difficult to interpret. One reason is that quantitative literacy has no fixed standard against which it can be measured by a test. What seems beyond doubt, however, is that, as a nation the United States is most definitely not quantitatively literate. I believe that this deficiency is a result of our (1) not regarding quantitative literacy as a basic responsibility of *all* teachers, on a par with basic language skills, (2) confusing quantitative literacy with mathematics, and (3) chasing the goal of achieving widespread proficiency in parts of *mathematics*, which for all but a small minority is simply unattainable (and, contrary to an oft expressed opinion that the decline in mathematics skills is a recent phenomenon, was probably not attained in the past, either).

Many more people appreciate music than can play a musical instrument. Many more people can enjoy a good novel or play than could write one. Many more people enjoy the benefits of driving an automobile than have the knowledge or skill to repair one. Similarly, we should recognize that it is possible to help people appreciate mathematics without forcing them (in vain) to try to achieve a working skill in the discipline. If we recognize that quantitative literacy and mathematics are different, if we accept that quantitative literacy is everybody's responsibility, and if we teach mathematics with the goal of developing an awareness of, and an appreciation for, its nature, extent, and relevance to modern life, then I see no reason why we cannot make a dramatic improvement in both the overall level of quantitative literacy and the presently poor level of general mathematical (and scientific) literacy in the population at large.

HOW DO WE DO IT?

Providing students with the broad overview of mathematics I am advocating will require a major change in the way we prepare high school mathematics teachers. At the same time, ensuring that we do not in the process lose the equally important goal of quantitative literacy requires that we change the way we prepare *all* high school teachers, not just the mathematics and science teachers.

Providing a broad overview of mathematics to today's students will require the imaginative use of all available media, in particular video and interactive computer technologies, as well as the preparation of first-rate printed material. I am avoiding using the word *textbooks*, since I do not believe that textbooks are suitable for the kind of instruction I am suggesting. Far better are high-quality expository books of the kind usually referred to as *popular science books*. I list a number of such books at the end of this article.

In many ways, what I am advocating does not require the development of a radical new way of teaching. Teachers of history, geography, economics, and psychology know how to teach subjects that have both descriptive, factual content and procedural aspects. Indeed, in those disciplines, the teaching is generally more factual and descriptive than procedural. My suggestion really amounts to adopting a similar style of teaching for mathematics, though perhaps with a different balance between the descriptive and the procedural elements.

Of course, mathematics is too extensive to expect any teacher to be able to master it all. All of mathematics, however, consists of variations on the same theme: the identification, abstraction, study, and application of patterns, using the mental tools of logical reasoning. A teacher who has mastered one area of mathematics will have little difficulty guiding students on a voyage of discovery in any other branch, provided that the experts in those other areas produce sufficiently accessible expository materials. An example of the kinds of material I have in mind is the 1998 Public Broadcasting Service production *Life by the Numbers*. Produced by WQED Television in Pittsburgh, this seven-part television series, also available on videocassette, showed the enormous extent of modern mathematics and the role it plays in all aspects of our lives.

In the end, however, education is about people. No matter how much money we spend on stimulating books, glossy television programs, and fancy computer products, we will not make any real progress unless we channel far greater funds into teacher preparation than we have in the recent past. That issue is societal. I hope it is not too idealistic a dream to hope that by restructuring grades K–12 mathematics education to provide a broad overview of the nature and breadth of mathematics and the role it plays in modern society, we will produce a new generation of public leaders who are themselves sufficiently appreciative of mathematics to recognize the importance of developing an adequate supply of well-prepared mathematics teachers, equipped with the resources necessary to give our young people the mathematics education they need and deserve. As I observed earlier, mathematics is the fuel that drives the information age. And education is the way we manufacture that fuel.

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