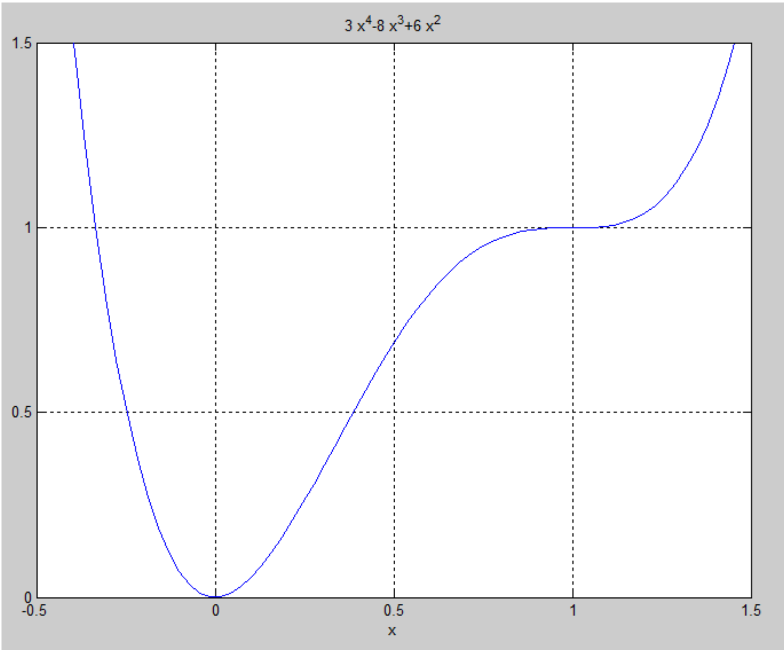
MA 131 – Sketches of solutions for some practice exam problems

1. Function is $f(x) = 3x^4 - 8x^3 + 6x^2$, so derivative is $f'(x) = 12x^3 - 24x^2 + 12x = 12x(x^2 - 2x + 1)$, which factors further: $f'(x) = 12x(x-1)(x-1)$. For critical numbers, this is zero when x = 0, 1. Make a number line, and test values like –1, 0.5, and 2. We find that the curve is decreasing, then increasing, then increasing, so there's a local minimum at x = 0 – the point is (0, 0). There is no local maximum. Now, we take the second derivative, which is $f''(x) = 36x^2 - 48x + 12 = 12(3x^2 - 4x + 1)$, and this factors, so $f''(x) = 12(3x-1)(x-1)$, and this is zero when $x = \frac{1}{3}, 1$. Make <u>another</u> number line, and test values such as 0, 0.5, and 2. We find that the function is concave up in the first interval, concave down in the second interval, and concave up in the third. So, the points (0, 0) and (1, 1) are points of inflection. The only intercept is (0, 0). Here's the curve's actual sketch, in the areas of importance:



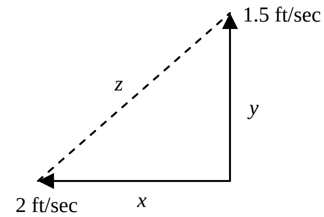
2. For the function $y = x^{5x^2}$, we use logarithmic differentiation: $\ln y = \ln x^{5x^2} \Rightarrow \ln y = (5x^2)\ln x$, and then we differentiate implicitly. $\frac{1}{y}\frac{dy}{dx} = (5x^2)\left(\frac{1}{x}\right) + (10x)\ln x$ (the right side required the product rule). So, solving, we get $\frac{dy}{dx} = y(5x + 10x\ln x) = x^{5x^2}(5x + 10x\ln x)$, and we're done.

3. Left to right: $8y^3\frac{dy}{dx} - 3\left(x\frac{dy}{dx} + y\right) = 1 + 5\frac{dy}{dx} \Rightarrow (8y^3 - 3x - 5)\frac{dy}{dx} = 1 + 3y \Rightarrow \frac{dy}{dx} = \frac{1+3y}{8y^3 - 3x - 5}$.

4. $\frac{dy}{dx} = 5(\cot(4x^3))^4(-\csc^2(4x^3))(12x^2) = -60x^2\cot^4(4x^3)\csc^2(4x^3)$.

5. $\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x-1)^2}}(2) = \frac{2}{\sqrt{1-(4x^2-4x+1)}} = \frac{2}{\sqrt{4x-4x^2}} = \frac{2}{\sqrt{4x(1-x)}} = \frac{1}{\sqrt{x(1-x)}}$.

6. Product rule: $\frac{dy}{dx} = \sin x\left(\frac{1}{x^2}(2x)\right) + \cos x\left(\ln\left(x^2\right)\right) = \frac{2\sin x}{x} + \ln\left(x^2\right)\cos x$.



7. A diagram is to the right. Let x be the distance Phyllis has walked from the start, and let y be the distance Joe has walked (both in feet). Then, let z be the distance between Joe and Phyllis. Our equation is $x^2 + y^2 = z^2$, and we want to find $\frac{dz}{dt}$ when x = 10 ft. If Phyllis has walked 10 feet, then it has taken 5 seconds, so Joe has walked 7.5 feet. From these, we can use our equation (the Pythagorean theorem, naturally) to find z = 12.5 feet. We also know that $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 1.5$. Now, we differentiate our equation with respect to time t, and that gives us $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$. We know five of these quantities, and want to find the sixth. Plug in: $2(10)(2) + 2(7.5)(1.5) = 2(12.5)\frac{dz}{dt}$, which gives us $40 + 22.5 = 25\frac{dz}{dt} \Rightarrow \frac{dz}{dt} = \frac{62.5}{25} = \frac{5}{2}$, so the distance between them is increasing at 2.5 ft/sec.

8. $L(x) = f(a) + f'(a)(x-a)$, and $f(a) = f(2) = 21$. We take the derivative, so $f'(x) = 9x^2 - 2x$, and $f'(a) = f'(2) = 32$. So, our formula is $L(x) = 21 + 32(x-2) = 32x - 43$.

9. Find the endpoints of the function on the interval: $f(-2) = -28$ and $f(1) = 2$. So, the (average) slope – found by using the slope formula on the endpoints – is $m = \frac{2+28}{1+2} = \frac{30}{3} = 10$. The derivative is $f'(x) = 9x^2 - 2x$, so we need to set this equal to 10 and solve: $9x^2 - 2x = 10 \Rightarrow 9x^2 - 2x - 10 = 0$. The quadratic formula gives us $x = \frac{2 \pm \sqrt{4 - 4(9)(-10)}}{18} = \frac{2 \pm \sqrt{364}}{18} = \frac{1 \pm \sqrt{91}}{9}$. The one of these that's in the interval we have is $c = \frac{1-\sqrt{91}}{9}$. *This was made up quickly – if I give you an MVT problem on the test, it will be likely more "friendly."*

10. Similar to #9, except the slope *should* come out to be zero (I'm actually going to show that). Find the endpoints: $f(0) = 0$ and $f(3) = 27 - 3(9) = 0$, so the slope is 0. The derivative is $f'(x) = 3x^2 - 6x$, so we set this equal to 0, and get $3x(x-2) = 0 \Rightarrow x = 0,\ 2$. Our value of c = 2 (the one in the interval).

11. Since y = 4, that means $f(x) = 4$, so $x^2 - 3x = 4 \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x-4)(x+1) = 0$, which gives us x = –1, 4. The leftmost location is x = –1. Derivative is $f'(x) = 2x - 3$, so the slope at x = –1 is simply $f'(-1) = -5$.

I did these VERY quickly, so if you think there's a mistake, please send me an email and I'll check it out!