Limit Samples

$\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}$ There was a typo in the original problem!

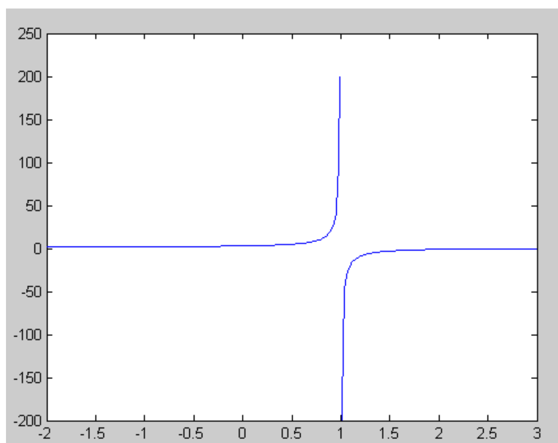
For this we use the conjugate of the numerator: $\lim_{x\to -4}\left(\frac{\sqrt{x^2+9}-5}{x+4}\right)\left(\frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5}\right)$. Multiply out the numerator, but NOT the denominator: $\lim_{x\to -4}\frac{(x^2+9-25)}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{x^2-16}{(x+4)\left(\sqrt{x^2+9}+5\right)}$. Now we can factor the numerator: $\lim_{x\to -4}\frac{(x-4)(x+4)}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{x-4}{\left(\sqrt{x^2+9}+5\right)}=\frac{-8}{10}=-\frac{4}{5}$.

$\lim_{x\to -1}\frac{x^2-1}{2x-2}=\frac{0}{-4}=0$ Don't overlook being able to just plug in to find the limit value!

$\lim_{h\to\infty}\frac{\sqrt{9h^6-h}}{h^3+1}$

For this, use the reciprocal of the highest power of h found in the denominator, multiplied on top and bottom: $\lim_{h\to\infty}\left(\frac{\sqrt{9h^6-h}}{h^3+1}\right)\left(\frac{\frac{1}{h^3}}{\frac{1}{h^3}}\right)=\lim_{h\to\infty}\frac{\sqrt{\frac{1}{h^6}}\sqrt{9h^6-h}}{1+\frac{1}{h^3}}=\lim_{h\to\infty}\frac{\sqrt{9-\frac{1}{h^5}}}{1+\frac{1}{h^3}}=3$. Note that each term involving a variable denominator will tend toward zero as h approaches infinity.

$\lim_{x\to 1^+}\frac{x^2-9}{x^2+2x-3}=\lim_{x\to 1^+}\frac{(x-3)(x+3)}{(x+3)(x-1)}=\lim_{x\to 1^+}\frac{x-3}{x-1}$. Note that this does not allow us to plug in, but it does make it easier for us to evaluate the limit. When x is approaching 1 *from the right*, the numerator of this expression is approaching –2 (also from the right, but that doesn't really matter), while the denominator is approaching 0 *from the right*, meaning it's positive. So, this is an infinite limit, and will tend toward $-\infty$, since the numerator is negative and the denominator is positive. Here's a sketch of the graph to show the trend – there's also a "hole" in the graph at $x = -3$ (a removable discontinuity).



$\lim_{u\to 1}\frac{u^4-1}{u^3+5u^2-6u}=\lim_{u\to 1}\frac{(u^2-1)(u^2+1)}{u(u^2+5u-6)}=\lim_{u\to 1}\frac{(u-1)(u+1)(u^2+1)}{u(u+6)(u-1)}=\lim_{u\to 1}\frac{(u+1)(u^2+1)}{u(u+6)}=\frac{(2)(2)}{1(7)}=\frac{4}{7}$.

Derivative Samples:

$y=\dfrac{3t-5t^2\sqrt{t}}{\sqrt{t}}=\dfrac{3t}{t^{1/2}}-\dfrac{5t^2\left(t^{1/2}\right)}{t^{1/2}}=3t^{1/2}-5t^2$, so $\dfrac{dy}{dt}=3\left(\dfrac{1}{2}t^{-1/2}\right)-10t=\dfrac{3}{2\sqrt{t}}-10t$. If you try to do this using the quotient rule, shame on you!! ☺ It makes the problem much longer, which will take more time than expected…

$f(x)=5e^{2x}\csc(x)$, so we'll use product and chain rules: $f'(x)=\left(5e^{2x}\right)\left(-\csc x\cot x\right)+\left(\csc x\right)\left(5e^{2x}\right)(2)$. Simplifying doesn't give much more: $f'(x)=-5e^{2x}\csc x\cot x+10e^{2x}\csc x$. This could be written factored, as $f'(x)=5e^{2x}\csc x\left[2-\cot x\right]$.

$g(x)=\dfrac{x^2-x}{3x^2+5}$. Quotient rule is required: $g'(x)=\dfrac{\left(3x^2+5\right)(2x-1)-\left(x^2-x\right)(6x)}{\left(3x^2+5\right)^2}$. Cleaning this up gives us: $g'(x)=\dfrac{6x^3-3x^2+10x-5-6x^3+6x^2}{\left(3x^2+5\right)^2}=\dfrac{3x^2+10x-5}{\left(3x^2+5\right)^2}$.

$y=3\sin\left(\dfrac{5\pi}{6}\right)$. This is just a constant, so the derivative of a constant is 0! Don't be tricked by one of these!!

$x^3+y^3=3$, done implicitly. So, we get $3x^2+3y^2\dfrac{dy}{dx}=0\Rightarrow 3y^2\dfrac{dy}{dx}=-3x^2\Rightarrow\dfrac{dy}{dx}=-\dfrac{x^2}{y^2}$. *An implicit differentiation problem on your test will probably be a relatively simple one, since it's a "fresh" topic.*

$f(x)=\dfrac{3}{x}$ using the definition. Okay, so $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{\dfrac{3}{x+h}-\dfrac{3}{x}}{h}$. We need to combine the two fractions in the numerator: $\lim\limits_{h\to 0}\dfrac{\left(\dfrac{x}{x}\right)\left(\dfrac{3}{x+h}\right)-\left(\dfrac{3}{x}\right)\left(\dfrac{x+h}{x+h}\right)}{h}=\lim\limits_{h\to 0}\dfrac{\dfrac{3x-3x-3h}{x(x+h)}}{h}$, which gives us $\lim\limits_{h\to 0}\dfrac{\dfrac{-3h}{x(x+h)}}{\dfrac{h}{1}}=\lim\limits_{h\to 0}\left(\dfrac{-3h}{x(x+h)}\right)\left(\dfrac{1}{h}\right)=\lim\limits_{h\to 0}\dfrac{-3}{x(x+h)}=\dfrac{-3}{x^2}$. *Of course, we could check this because we know how to do it using the power rule.*

Continuity Sample:

$$f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x^2 - 4x + 3} & \text{if } x < 3 \\ mx + 4 & \text{if } x \geq 3 \end{cases}$$

I factor the numerator and denominator of the first part: $\dfrac{(2x+1)(x-3)}{(x-3)(x-1)}$. This tells me the function is not defined at $x = 1$, so it cannot be continuous there. This is an <u>infinite discontinuity</u>. We also have to determine what value of m will "force" continuity at $x = 3$, another place where there could be a problem. To find this, we must do limits from the left and the right:

$\lim\limits_{x \to 3^-} \dfrac{2x+1}{x-1} = \dfrac{7}{2}$, and $\lim\limits_{x \to 3^+} (mx+4) = 3m+4$. These must be equal, so $\dfrac{7}{2} = 3m + 4 \Rightarrow 3m = -\dfrac{1}{2}$, and so $m = -\dfrac{1}{6}$. (Note that if m were any <u>other</u> value, we'd also have a <u>jump discontinuity</u> at $x = 3$.)

Word Problem Samples:

To write a tangent line for $y = \frac{2x}{(x+1)^2}$, we need a point and the slope (derivative). We're given $x = 1$, so we can find the point: $y = \frac{2(1)}{(1+1)^2} = \frac{1}{2}$, so our point is $\left(1, \frac{1}{2}\right)$. To find the slope, let's take the derivative: $\frac{dy}{dx} = \frac{(x+1)^2(2) - 2x\left(2(x+1)\right)}{(x+1)^4}$. *We really don't need to clean this up much, let's just plug in the x = 1*. So, our slope is $\frac{dy}{dx} = \frac{2^2(2) - 2(2(2))}{2^4} = \frac{0}{16} = 0$. So, our tangent line is horizontal, and its equation is just $y = \frac{1}{2}$.

$V(t) = 36 - 24t + 4t^2$

First question asks about a rate (derivative): $V'(t) = -24 + 8t$. Plug in $t = 2$, and we get –8. But, the problem says "draining," so we want our answer to be positive: **8 gallons per minute**. *You must include the units!*

How much water drains in the first two minutes? Well, when $t = 0$, $V = 36$. When $t = 2$, $V = 36 - 48 + 16 = 4$. So, **32 gallons** have drained out in the first two minutes.

The average rate at which the water has drained out in the first two minutes is 32/2 = **16 gallons per minute**.

Working with a Graph:

The first question asks for any places where the graph is not differentiable. For differentiability, the graph basically needs to be "smooth." It is clear from the graph that at $x = -1$, the graph is not differentiable.

For the other sketches, I've drawn them and scanned the results in on the following page. Here are the explanations:

i. this is just the original graph shifted to the right 1 unit (horizontal translation)

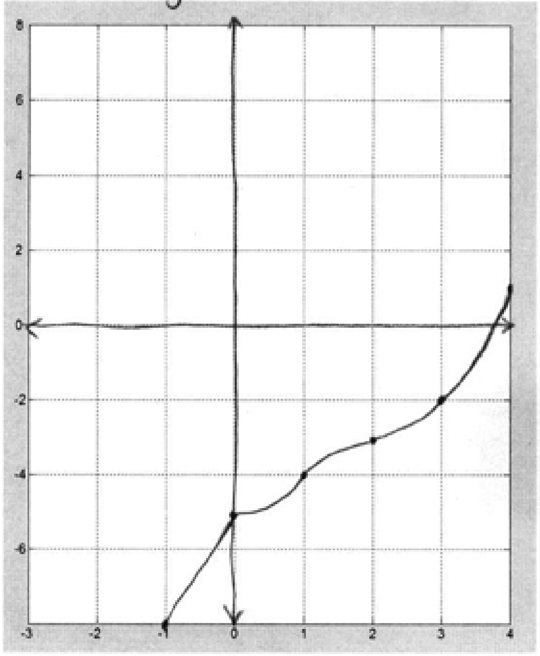
ii. this is the original graph reflected through the *y*-axis

iii. this graph is the original compressed in toward the *y*-axis – every point is half as far from the *y*-axis as it was originally

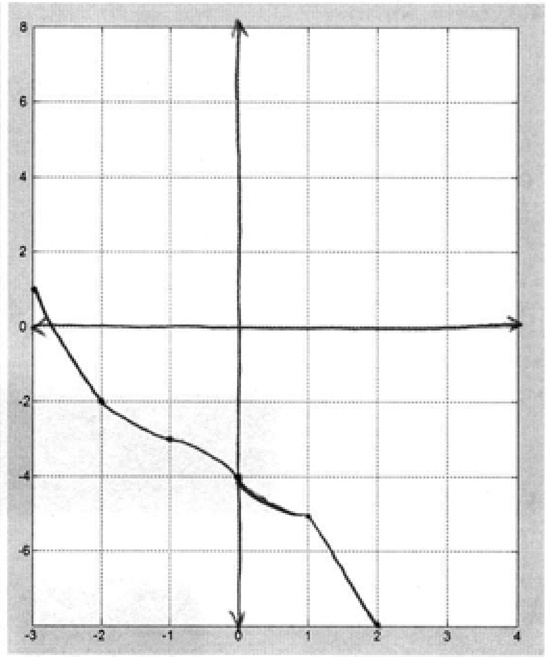
iv. this sketch will need to be the slope of the tangent line at every point on the original graph – this is a little tricky, and we need to estimate some of these slopes, but I think my sketch is pretty close. I started by finding the (constant) slope of the linear section. Then, it seemed that the right hand piece of the graph (the curvy part) started with a horizontal tangent, which means the slope would start at zero. Since this part goes uphill the whole way, we know the slope will always be positive, so the derivative will be above the *y*-axis. However, the slopes increase for a little while, then decrease briefly, then increase again for as long as we can see the graph. To be more accurate, I'd really need to draw some sample tangents with a ruler, pick points on them, and calculate the slopes using the slope formula. I'm too tired… ☺

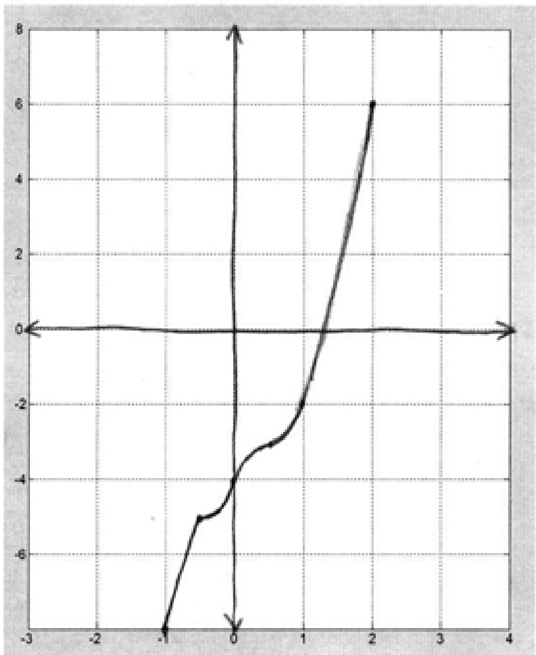
The last question here asks for an inverse function value. So, the –2 that they give us is a *y*-value, and so we go to –2 on the *y*-axis, move over until we find the point on the curve, and find that the *x*-value at that point is 2. We determine that $f^{-1}(-2) = 2$.

b(i) $g(x) = f(x-1)$

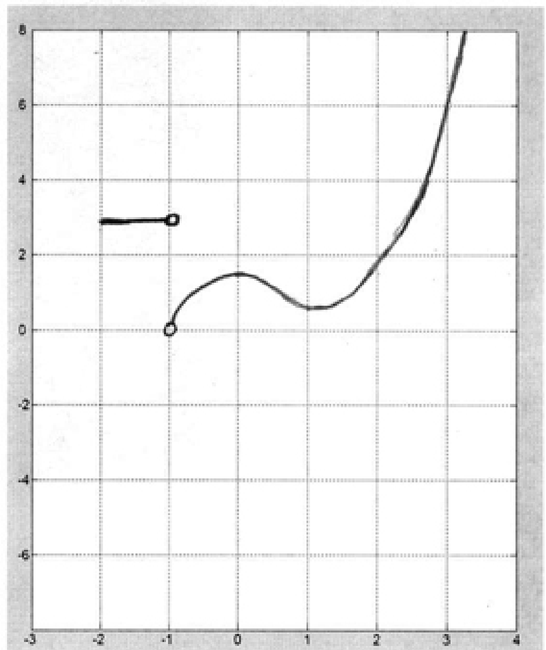


b(ii) $h(x) = f(-x)$





b(iii) $k(x) = f(2x)$



b(iv) $f'(x)$

$\left.\begin{array}{l}(-2,-8)\\(-1,-5)\end{array}\right\}$ linear part → slope is $\frac{-5 - -8}{-1 - -2} = \frac{3}{1} = 3$

no derivative at $x = -1$

on the right part, the slope appears to start at zero (horizontal tangent)

Also, recall that trig functions, especially the inverse trig functions, were not tested on the mini-chapter 1 exam, so make sure that you know how to deal with trigonometry well! Study the identities that exist between the trig functions, as given on a recent handout.