Let's solve the following equation: $y''-4y'+4y = 4\sin t + e^{2t}$.

First, we look at the corresponding homogeneous equation, which is $y''-4y'+4y = 0$ (*this is important to avoid problems later in the process*). This has characteristic equation $r^2 - 4y + 4 = 0$, and characteristic roots of r = 2 (repeated). So, the general solution is clearly $y_h(t) = c_1e^{2t} + c_2te^{2t}$.

Now, we must find a particular solution to the original equation; the "forcing function" is actually composed of a linear combination of functions from two different families. That is, we can consider $y''-4y'+4y = 4\sin t$ and $y''-4y'+4y = e^{2t}$ separately, using the method of undetermined coefficients for each piece.

For the first equation, we would believe that a solution would have to be in the form $y_{p_1} = A\sin t + B\cos t$. Then, $y_{p_1}' = A\cos t - B\sin t$ and $y_{p_1}'' = -A\sin t - B\cos t$. Substituting into our original (part 1) equation, we get $-A\sin t - B\cos t - 4(A\cos t - B\sin t) + 4(A\sin t + B\cos t) = 4\sin t$. Restructuring the left-hand side, we obtain $\sin t(-A + 4B + 4A) + \cos t(-B - 4A + 4B) = 4\sin t$, so $3A + 4B = 4$ and $-4A + 3B = 0$. Solving this little system yields $A = \frac{12}{25}$ and $B = \frac{16}{25}$. So, our first-part solution will be $y_{p_1} = \frac{12}{25}\sin t + \frac{16}{25}\cos t$.

Now, let's look at the second part: $y''-4y'+4y = e^{2t}$. We might think that our solution should be of the form $y_{p_2} = Ce^{2t}$, so $y_{p_2}' = 2Ce^{2t}$ and $y_{p_2}'' = 4Ce^{2t}$. However, when we substitute this in, here's what we get: $4Ce^{2t} - 8Ce^{2t} + 4Ce^{2t} = e^{2t} \Rightarrow 0 = e^{2t}$. Why does this happen? Well, it occurs because we already know (from our characteristic equation work) that any scalar multiple of e^{2t} will in fact be a solution to the corresponding homogeneous equation! So does this mean the Method of Undeteremined Coefficients won't work? No, it simply means we have to be more careful in choosing our function to use in the method. As before, perhaps we'd like to try, instead, $y_{p_2} = Cte^{2t}$. Will this have any better success?? Let's check it out…

$y_{p_2}' = 2Cte^{2t} + Ce^{2t}$, and $y_{p_2}'' = 4Cte^{2t} + 2Ce^{2t} + 2Ce^{2t} = 4Cte^{2t} + 4Ce^{2t}$. So, subbing in yields the following: $4Cte^{2t} + 4Ce^{2t} - 4(2Cte^{2t} + Ce^{2t}) + 4Cte^{2t} = e^{2t} \Rightarrow 0 = e^{2t}$. Crap! What's wrong here?

We should notice that, in reality, we have just tried the second piece of our known homogeneous solution, so we shouldn't be at all surprised that the left side turned out to be zero. So what are we to do?

Well, we continue to add another factor of t until we get past the number required by repeated real solutions to the characteristic equation for the corresponding homogeneous equation. Therefore, we should be able to use $y_{p_2} = Ct^2e^{2t}$ and apply the Method of Undetermined Coefficients successfully. Let's try…

$y_{p_2}' = 2Ct^2e^{2t} + 2Cte^{2t}$ and $y_{p_2}'' = 4Ct^2e^{2t} + 4Cte^{2t} + 4Cte^{2t} + 2Ce^{2t} = 4Ct^2e^{2t} + 8Cte^{2t} + 2Ce^{2t}$. So, subbing into our equation now gives $4Ct^2e^{2t} + 8Cte^{2t} + 2Ce^{2t} - 4(2Ct^2e^{2t} + 2Cte^{2t}) + 4(Ct^2e^{2t}) = e^{2t}$. Collecting all the left side terms, since each has the common factor, yields $e^{2t}(4Ct^2 + 8Ct + 2C - 8Ct^2 - 8Ct + 4Ct^2) = e^{2t}$, and so $e^{2t}(t^2(4C - 8C + 4C) + t(8C - 8C) + 2C) = e^{2t}$, so clearly $2C = 1$, and therefore $C = \frac{1}{2}$. So, our second part solution is $y_{p_2} = \frac{1}{2}t^2e^{2t}$. Now, let's put this all together!

Since $y_{p_1} = \frac{12}{25}\sin t + \frac{16}{25}\cos t$ is a particular solution of $y''-4y'+4y = 4\sin t$, and $y_{p_2} = \frac{1}{2}t^2 e^{2t}$ is a particular solution to $y''-4y'+4y = e^{2t}$, we know from the Superposition Principle that the sum of these two functions, $y_p = \left(\frac{12}{25}\sin t + \frac{16}{25}\cos t\right) + \left(\frac{1}{2}t^2 e^{2t}\right)$ is a particular solution to $y''-4y'+4y = 4\sin t + e^{2t}$, and so we have our single, particular solution. We already had solved the corresponding homogeneous equation previously (that was the <u>easy</u> part!), finding $y_h(t) = c_1 e^{2t} + c_2 t e^{2t}$. Finally, from the Non-Homogeneous Principle, we have our general solution to this DE:

$$y(t) = y_h + y_p = c_1 e^{2t} + c_2 t e^{2t} + \left(\frac{12}{25}\sin t + \frac{16}{25}\cos t\right) + \left(\frac{1}{2}t^2 e^{2t}\right).$$

Wow…