Sample Final Problem 1

$x''-4x'-5x=0$ – change it to a system and solve it using chapter 6 techniques…

Converting to a system ~10 points

Let $y=x'$, which naturally means $y'=x''$.

Substitute into the original equation to get $y'-4y-5x=0$.

So, here's our system of two first-order equations: $\begin{aligned} x' &= y \\ y' &= 5x+4y \end{aligned}$.

We write this in matrix-vector form so we can apply techniques from chapter 6: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. If we let the vector $\vec{\mathbf{x}}=\begin{bmatrix} x \\ y \end{bmatrix}$, and matrix $\mathbf{A}=\begin{bmatrix} 0 & 1 \\ 5 & 4 \end{bmatrix}$, then the equation can be written as $\vec{\mathbf{x}}'=\mathbf{A}\vec{\mathbf{x}}$.

Solving the system ~10 points

To solve the system, we know that the first step is to find eigenpairs. So, I'll find the eigenvalues for matrix **A**.

$\begin{vmatrix} -\lambda & 1 \\ 5 & 4-\lambda \end{vmatrix}=0$, so $\lambda^2-4\lambda-5=0 \Rightarrow (\lambda-5)(\lambda+1)=0$, and therefore eigenvalues are $\lambda_1=-1$ and $\lambda_2=5$.

Now, I will find an eigenvector to pair with $\lambda_1=-1$. To do this, I must solve $\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (If I were to augment this system and put it in RREF, I'd have $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 5 & 5 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$.) So, this means that $1v_1+1v_2=0$, or $v_1=-v_2$; I pick something simple like $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (any vector having opposite entries would be fine).

Finding an eigenvector to pair with $\lambda_2=5$ is an identical process. $\left[\begin{array}{cc|c} -5 & 1 & 0 \\ 5 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -1/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$, so $-5v_1+v_2=0 \Rightarrow v_2=5v_1$. I select $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

My general solution to the system is $\vec{\mathbf{x}}=c_1e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}+c_2e^{5t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Using the initial conditions vector to find the final answer to the IVP ~5 points

With the initial condition as given, $\vec{\mathbf{x}}(\mathbf{0})=\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, we can substitute in to obtain $\begin{bmatrix} 2 \\ 0 \end{bmatrix}=c_1e^0\begin{bmatrix} 1 \\ -1 \end{bmatrix}+c_2e^0\begin{bmatrix} 1 \\ 5 \end{bmatrix}$, which gives us $c_1+c_2=2$ and $-c_1+5c_2=0$. We can easily solve this to obtain $c_2=\frac{1}{3}$ and $c_1=\frac{5}{3}$. The final solution to the IVP is therefore $\vec{\mathbf{x}}=\frac{5}{3}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}+\frac{1}{3}e^{5t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.