Sample Final Problem 2

Solving $y''+y'-12y=65\sin 2t$, with $y(0)=-1$ and $y'(0)=1$

a. using methods of chapter 4…

<u>Solve the corresponding homogeneous solution</u> (to find y_h) ~5 points

$y''+y'-12y=0$, so I'll write and solve the characteristic equation: $r^2+r-12=0 \Rightarrow (r+4)(r-3)=0$, which gives me $r = -4$ and $r = 3$. Thus, my general solution to the corresponding homogeneous equation is the equation $y_h = c_1e^{-4t}+c_2e^{3t}$.

<u>Find a particular solution to the original DE</u> (this was called y_p) ~10 points

I will use the Method of Undetermined Coefficients for this part. So, I suppose that $y_p = A\cos 2t + B\sin 2t$. That means $y_p' = -2A\sin 2t + 2B\cos 2t$ and $y_p'' = -4A\cos 2t - 4B\sin 2t$. Plugging these into the original equation yields $-4A\cos 2t - 4B\sin 2t + (-2A\sin 2t + 2B\cos 2t) - 12(A\cos 2t + B\sin 2t) = 65\sin 2t$. Now, when I rearrange these terms on the left, I get $(-4A+2B-12A)\cos 2t + (-4B-2A-12B)\sin 2t = 65\sin 2t$. This results in the simple system: $-16A+2B=0$ and $-2A-16B=65$. I multiply the first equation by 8, which gives $-128A+16B=0$, and then I add this to the second equation, $-130A=65$, so $A=-\frac{1}{2}$. Then, it's easy to see that $B = -4$. From this work, I know that a particular solution is $y_p = -\frac{1}{2}\cos 2t - 4\sin 2t$, and a general solution to the whole DE is $y(t) = c_1e^{-4t}+c_2e^{3t}-\frac{1}{2}\cos 2t - 4\sin 2t$.

<u>Determine the arbitrary constants using initial conditions</u> ~10 points

I already know the general equation $y(t)$, and since $y(0)=-1$, I substitute in: $-1 = c_1 + c_2 - \frac{1}{2}$, which can simplify to $c_1 + c_2 = -\frac{1}{2}$. Next, I find the derivative $y' = -4c_1e^{-4t}+3c_2e^{3t}+\sin 2t - 8\cos 2t$, and since $y'(0)=1$, I know that $1 = -4c_1 + 3c_2 - 8$, or $-4c_1+3c_2=9$. I multiply the first equation in my system by 4, which gives $4c_1+4c_2=-2$, and then I add that to the second equation: $7c_2=7$, so $c_2=1$. It is then easy to see that $c_1 = -\frac{3}{2}$. My final answer to the IVP is $y(t) = -\frac{3}{2}e^{-4t}+e^{3t}-\frac{1}{2}\cos 2t - 4\sin 2t$.

b. using the LaPlace transform and its inverse…

Take the transform of both sides ~10 points

$\mathcal{L}\{y''+y'-12y\}=\mathcal{L}\{65\sin 2t\}$, so $\mathcal{L}\{y''\}+\mathcal{L}\{y'\}-12\mathcal{L}\{y\}=65\mathcal{L}\{\sin 2t\}$ by linearity. Now, we apply the principles as to how the transform deals with derivatives, and evaluate the right side using the table: $s^2\mathcal{L}\{y\}-s\,y(0)-y'(0)+(s\,\mathcal{L}\{y\}-y(0))-12\mathcal{L}\{y\}=65\left(\frac{2}{s^2+4}\right)$. On the left side, we know that $y(0)=-1$ and $y'(0)=1$, so we substitute those in: $s^2\mathcal{L}\{y\}+s-1+s\,\mathcal{L}\{y\}+1-12\mathcal{L}\{y\}=65\left(\frac{2}{s^2+4}\right)$. Now, I can simply collect and factor the terms involving the transform, $\mathcal{L}\{y\}(s^2+s-12)+s=65\left(\frac{2}{s^2+4}\right)$, or equivalently $\mathcal{L}\{y\}(s^2+s-12)=65\left(\frac{2}{s^2+4}\right)-s$. Before I continue with anything further, let me "clean up" the right side a bit… $65\left(\frac{2}{s^2+4}\right)-s=\frac{130}{s^2+4}+\frac{(-s)(s^2+4)}{s^2+4}$, which becomes $\frac{130-s^3-4s}{s^2+4}=\frac{-s^3-4s+130}{s^2+4}$. So, my equation above involving the transform is simply $\mathcal{L}\{y\}(s^2+s-12)=\frac{-s^3-4s+130}{s^2+4}$, which means $\mathcal{L}\{y\}=\frac{-s^3-4s+130}{(s^2+4)(s-3)(s+4)}$.

Find the solution using the inverse transform ~15 points

This is just going to be tedious algebra, but it's not terribly difficult. I'll use partial fraction separation on the transform $\frac{-s^3-4s+130}{(s^2+4)(s-3)(s+4)}=\frac{As+B}{s^2+4}+\frac{C}{s-3}+\frac{D}{s+4}$. Putting these "back together" is a bit of a pain, but yields $\frac{(As+B)(s-3)(s+4)+C(s^2+4)(s+4)+D(s^2+4)(s-3)}{(s^2+4)(s-3)(s+4)}$. I distribute out the numerator, then factor it, resulting in $\frac{(A+C+D)s^3+(A+B+4C-3D)s^2+(-12A+B+4C+4D)s+(-12B+16C-12D)}{(s^2+4)(s-3)(s+4)}$, and I can now establish my system: $A+C+D=-1$, $A+B+4C-3D=0$, $-12A+B+4C+4D=-4$, and finally, $-12B+16C-12D=130$. Multiply the first equation by –4: $-4A-4C-4D=4$, then add it to the third equation: $-16A+B=0$, so $B=16A$. Plug this into the second equation: $17A+4C-3D=0$, and the fourth equation: $-192A+16C-12D=130$, and combine with $A+C+D=-1$ so we now have a 3x3 system. Multiply this last equation by 3: $3A+3C+3D=-3$, add to $17A+4C-3D=0$, gives $20A+7C=-3$. Multiply $A+C+D=-1$ by 12: $12A+12C+12D=-12$, then add to $-192A+16C-12D=130$ to get $-180A+28C=118$, with $20A+7C=-3$ is 2x2 system. Multiply second equation by –4: $-80A-28C=12$, then add $-260A=130$, so $A=-\frac{1}{2}$, meaning $B=-8$. Also, $20A+7C=-3$ yields $-10+7C=-3$, or $C=1$. Finally, since $A+C+D=-1$, we get $D=-\frac{3}{2}$. Whew… Thus, our solution to the IVP is found as follows:

$y=\mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}s-8}{s^2+4}+\frac{1}{s-3}+\left(\frac{-\frac{3}{2}}{s+4}\right)\right\}=\mathcal{L}^{-1}\left\{-\frac{1}{2}\left(\frac{s}{s^2+4}\right)-4\left(\frac{2}{s^2+4}\right)+\left(\frac{1}{s-3}\right)-\frac{3}{2}\left(\frac{1}{s+4}\right)\right\}$, and these can all be picked up from the table now: $y(t)=-\frac{1}{2}\cos 2t-4\sin 2t+e^{3t}-\frac{3}{2}e^{-4t}$. *I'm really glad it worked out…* ☺