Sample Final Problem 5

Solving $\dot{\vec{\mathbf{x}}} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} \vec{\mathbf{x}}$ using techniques of chapter 6

I find eigenvalues for the matrix $\mathbf{A} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}$ from the characteristic equation: $\begin{vmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{vmatrix} = 0$. This gives me $(-2-\lambda)(-2-\lambda) - (-3)(3) = 0 \Rightarrow \lambda^2 + 4\lambda + 4 + 9 = 0 \Rightarrow \lambda^2 + 4\lambda + 13 = 0$. This quadratic is not factorable, so we apply the quadratic formula: $\lambda = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2(1)} = -2 \pm 3i$. When eigenvalues are complex, we quickly know several things – namely, that the trajectories are *spirals* about the equilibrium at the origin, and that the stability of the origin is only dependent on the *sign* of the real part of the complex eigenvalues. Since $\lambda = \alpha \pm \beta i = -2 \pm 3i$, we know that $\alpha = -2 < 0$, and so the equilibrium at the origin is <u>asymptotically stable</u>, and called an <u>attracting spiral</u> or <u>spiral sink</u>. Thus, just from knowing the eigenvalues, I am able to answer the final parts of the question. I still have to write the general solution, so I go on to find an eigenvector (remember I only need <u>one</u> eigenpair to find the general solution when eigenvalues are complex).

So I choose the eigenvalue $-2+3i$ and look for an eigenvector: $\begin{bmatrix} -2-(-2+3i) & -3 \\ 3 & -2-(-2+3i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\left[\begin{array}{cc|c} -3i & -3 & 0 \\ 3 & -3i & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$ in RREF. This gives the relationship $v_1 - iv_2 = 0 \Rightarrow v_1 = iv_2$, so I'll choose $\begin{bmatrix} i \\ 1 \end{bmatrix}$, which I must think of as $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Now, I simply have to identify all of the things I need: $\alpha = -2$, $\beta = 3$ (chosen positive for convenience, as always), $\vec{\mathbf{p}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (the real part of the eigenvector), $\vec{\mathbf{q}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (the imaginary part of the eigenvector).

General solution to the equation is $\vec{\mathbf{x}}(t) = c_1 e^{-2t} \left(\cos 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin 3t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + c_2 e^{-2t} \left(\sin 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos 3t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$.