Sample Test Problem Solutions – MA 232

1. $\dfrac{dA}{dt} = k\dfrac{H}{G}$

2. $a = \dfrac{dv}{dt} = -32.2 \Rightarrow dv = -32.2dt \Rightarrow \int dv = \int -32.2dt \Rightarrow v = -32.2t + C$. Since $v = 0$ when $t = 0$, we have $C = 0$, so $v = -32.2t$.

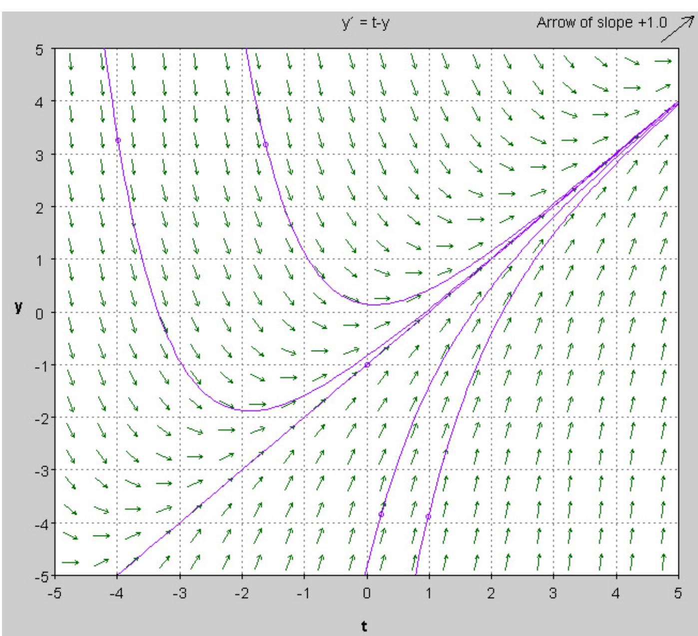
   $v = \dfrac{ds}{dt} = -32.2t \Rightarrow ds = -32.2t\,dt \Rightarrow \int ds = \int -32.2t\,dt \Rightarrow s = -16.1t^2 + C_s$. Since s (the height off the ground) is 1,454 when $t = 0$, C_s =1,454, so $s = -16.1t^2 + 1{,}454$.

   $s = 0$ when the marble hits the ground, so $-16.1t^2 + 1454 = 0 \Rightarrow t = 9.53$ seconds. Thus, $v(9.53)$ will give us the desired result: –32.2(9.53) = –306.96 $\dfrac{\text{ft}}{\text{sec}}$. *Answer* ***must be*** *negative since velocity was requested.*

   Note that under some circumstances, using the approximate gravitational constant of $-32\dfrac{\text{ft}}{\text{sec}^2}$ *will be sufficient. Be sure to read any problem involving gravity carefully, in case a particular value needs to be used.*

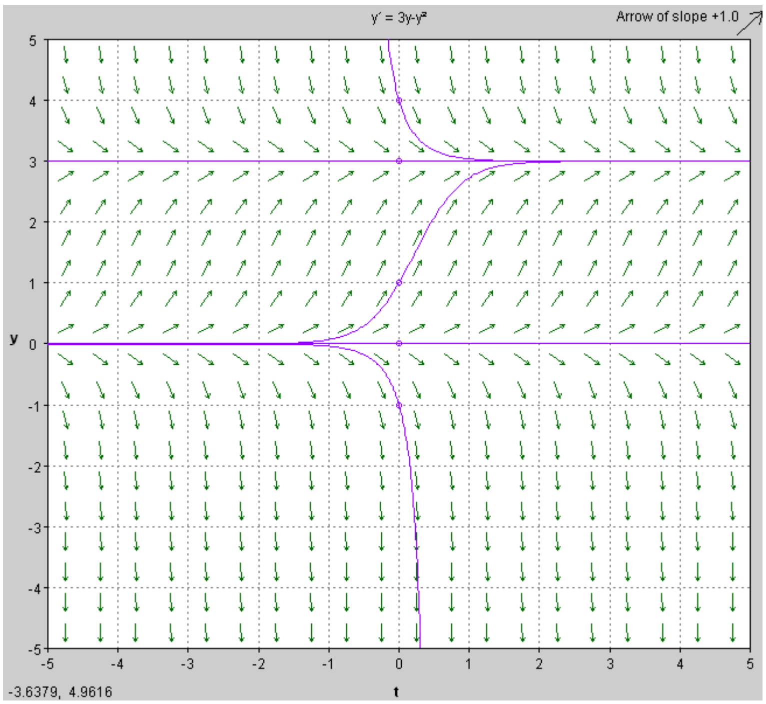
3. $y' = t - y$ has no equilibria, since $t - y = 0$ implies $y = t$, which depends on t. However, this can give us our first isocline, so along the line $y = t$, we can sketch direction field segments with a slope of zero. Other isoclines would also be parallel lines to this one, since setting $t - y = c$ will always yield the equation $y = t - c$. In fact, setting $t - y = 1$, which will yield the line $y = t - 1$, gives an actual solution to the DE (we can verify this quite easily).

   Sample solution curves that start above this linear solution decrease toward it, then turn and approach it asymptotically (always staying concave up). Sample solution curves that start below the linear solution head upward approaching it asymptotically, always staying concave down. A few possibilities are shown in the diagram below.



4. $y' = \dfrac{2t}{1+2y}$, $y(2) = 0$ – this is a separable DE. $(1+2y)\,dy = 2t\,dt \Rightarrow \int (1+2y)\,dy = \int 2t\,dt$, so we have $y + y^2 = t^2 + C$. Since we know the initial condition $y(2) = 0$, then $0 = 4 + C$, so $C = -4$. Thus, the solution is $y + y^2 = t^2 - 4$. *We can just leave this implicitly defined – don't get confused thinking you should solve it explicitly for y in terms of t!*

5. $y' = 3y - y^2$ has two equilibria: $y' = y(3-y)$, so $y = 0$ and $y = 3$ are both equilibria. A phase-line graph with testing in the three intervals of y will help to see stability information easily. As it turns out, if $y < 0$, we have negative slopes, if $0 < y < 3$ we have positive slopes, and if $y > 3$ we have negative slopes. Thus, $y = 0$ is unstable, and $y = 3$ is stable. This is shown in the diagram below, with a couple of sample solutions and the equilibrium solutions graphed.



6. We want to find the second derivative $\dfrac{d^2y}{dt^2}$ for the differential equation $y' = y^2 - 4t$. So, we must differentiate the equation using implicit differentiation and the chain rule as appropriate:

$y'' = 2y\left(\dfrac{dy}{dt}\right) - 4 = 2y\left(y^2 - 4t\right) - 4 = 2y^3 - 8yt - 4$. Now, $y'' = 0$ when $2y^3 - 8yt - 4 = 0$, which (solved for t for convenience, as mentioned in the directions) yields $t = \dfrac{2y^3 - 4}{8y} = \dfrac{y^3 - 2}{4y}$.

7. We need to perform Euler's method, with step size of 0.5, <u>twice</u> on the IVP $y' = \sqrt{t+y}$, with $y(1) = 3$. So, our starting point is (1, 3). Calculating the slope, we have $\sqrt{1+3} = 2$, so our slope is 2: "up 2, right 1," if you wish. However, we're using a step size of 0.5, so we'll only go "up 1, right 0.5," and our new point is (1.5, 4). Now, we repeat the process from this new location. Our new slope calculates to be $\sqrt{1.5+4} = \sqrt{5.5} \approx 2.3452$. Since our step size is 0.5, our second approximation point is (2, 5.1726). *I'd have to make it clear to what accuracy I wanted this answer, such as nearest tenth, etc.*

Now, we're also supposed to do this approximation with RK-2, which is just a little more complex. Initially, we do the same thing at (1, 3), calculating a slope of 2. But, instead of using this slope to take a step, we take a "pretend half-step" so we can calculate a modified slope. Our "pretend half-step" will be to $\left(\frac{5}{4}, \frac{7}{2}\right)$, so now we calculate a "usable" slope there: $\sqrt{\frac{5}{4}+\frac{7}{2}} = \sqrt{\frac{19}{4}} \approx 2.18$, so now we use that slope to take an actual step. We end up at (1.5, ~4.09). We repeat the process: find our (Euler) slope at this point, which is $\sqrt{1.5+4.09} = \sqrt{5.59} \approx 2.3643$, then take a "pretend half-step" to $\left(\frac{7}{4}, 4.681\right)$, and calculate a usable slope at that location: $\sqrt{1.75+4.681} = \sqrt{6.431} \approx 2.54$. Using this slope, we take our second (and final) step to arrive at (2, ~5.36). Whew… *Again, I'd have to make expected accuracy clear, or else make it so things worked our more cleanly!*

8. *a*. Second-order, linear (with variable coefficients), and homogeneous.
   b. Third-order, non-linear.

9. Solving $y' - y = e^{3t}$ with integrating factor method (recall the standard form of this equation is $y' + p(t)y = f(t)$). We must find $\mu(t) = e^{\int p(t)\,dt} = e^{\int -1 dt} = e^{-t}$, then multiply both sides by this integrating factor: $e^{-t}(y' - y) = e^{-t}e^{3t} \Rightarrow \frac{d}{dt}\left(ye^{-t}\right) = e^{2t} \Rightarrow d\left(ye^{-t}\right) = e^{2t}dt \Rightarrow \int d\left(ye^{-t}\right) = \int e^{2t}dt$, so $ye^{-t} = \frac{1}{2}e^{2t} + C \Rightarrow y = \frac{e^{2t}}{2e^{-t}} + \frac{C}{e^{-t}} \Rightarrow y = \frac{e^{3t}}{2} + Ce^{t}$. *If this were an IVP, we would use the initial condition to determine the value of C.*

Now, we're also asked to solve this using Euler-Lagrange process… To begin E-L, we must solve the associated homogeneous equation: $y' - y = 0$. This can be done easily by separation of variables, so $\frac{dy}{dt} = y \Rightarrow \int \frac{dy}{y} = \int dt \Rightarrow \ln|y| = t + C \Rightarrow y_h = C * e^{t}$. This is easily verified as a general solution. The second part of E-L requires us to suppose that $y_p = v(t)e^{t}$, and we can find $v(t)$ using the formula: $v(t) = \int f(t)\,e^{\int p(t)\,dt}\,dt = \int \left(e^{3t}\right)\left(e^{-t}\right)dt = \int e^{2t}\,dt = \frac{e^{2t}}{2}$ (no integration constant is needed here). Then, we can write our particular solution $y_p = \frac{e^{3t}}{2}$. Finally, we can write the general solution by adding these results: $y(t) = y_h + y_p = C * e^{t} + \frac{e^{3t}}{2}$. We can see that our results from each method agree.

10. $y' = ky \Rightarrow \frac{dy}{y} = k\,dt$, so after integrating both sides, we obtain $\ln|y| = kt + C$, which gives us $y = y_0 e^{kt}$, since we know that $y = y_0$ when $t = 0$. So, to find t_D, we solve $2y_0 = y_0 e^{kt_D}$ for t_D: $2 = e^{kt_D} \Rightarrow t_D = \frac{\ln 2}{k}$.

11. The formula for accumulation of continuously compounded interest is $A = A_0 e^{rt}$. We know $A_0 = 0.50$, and we know $r = 0.06$. Further, 160 years had passed, which is our t value (measured in years). So, we have $A = 0.50e^{(0.06)160} = 0.5e^{9.6} =$ 7,382.39. That's not bad from fifty cents!

12. Let x represent the amount of salt (in lbs) that is in the tank at any time t. Since we start with a tank (300 gals) of fresh water, we know that when $t = 0$, $x = 0$. We must determine "rate in" and "rate out" (of the salt) so that we can write our differential equation.

Rate in = (incoming concentration level) x (incoming flow rate) = 1 lb/gal x 3 gal/min = 3 lb/min

Rate out = (outgoing concentration level) x (outgoing flow rate) = $\dfrac{x}{300+2t}\dfrac{\text{lb}}{\text{gal}}$ x 1 gal/min = $\dfrac{x}{300+2t}\dfrac{\text{lb}}{\text{min}}$

So, our DE is $\dfrac{dx}{dt} = 3 - \dfrac{x}{300+2t}$, which can be rewritten as $\dfrac{dx}{dt} + \dfrac{1}{300+2t}x = 3$, so we recognize it as a linear, non-homogeneous first-order DE, which we can solve by integrating factor method.

$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{dt}{300+2t}} = e^{\frac{1}{2}\ln(300+2t)} = \sqrt{300+2t}$, so we multiply both sides by this:

$\sqrt{300+2t}\left(\dfrac{dx}{dt} + \dfrac{1}{300+2t}x\right) = 3\sqrt{300+2t}$. This problem is a little "less obvious" when it comes to converting the left side to a single derivative, but we can multiply it (the left side) out if we need to check on details: $\sqrt{300+2t}\,\dfrac{dx}{dt} + \dfrac{1}{\sqrt{300+2t}}x = 3\sqrt{300+2t}$. *For those who have discovered "the trick," this one follows the same pattern as all the others, of course.* Now, thinking of the product rule (backwards), we should be able to see that we have $\dfrac{d}{dt}\left(x\sqrt{300+2t}\right) = 3\sqrt{300+2t}$ or $d\left(x\sqrt{300+2t}\right) = 3\sqrt{300+2t}\,dt$. We can now integrate both sides: $x\sqrt{300+2t} = (300+2t)^{3/2} + C$, so $x(t) = 300 + 2t + \dfrac{C}{\sqrt{300+2t}}$. Since we know that $x = 0$ when $t = 0$, C is quickly found:

$0 = 300 + \dfrac{C}{\sqrt{300}} \Rightarrow C = -3000\sqrt{3}$, and our completed equation is $x(t) = 300 + 2t - \dfrac{3000\sqrt{3}}{\sqrt{300+2t}}$. Now, since the net increase in solution in the tank is 2 gal/min, it will take 150 minutes to fill the tank to the top, so we want to find the salt content when $t = 150$.

$x(150) = 300 + 300 - \dfrac{3000\sqrt{3}}{\sqrt{600}} = 600 - \dfrac{3000\sqrt{3}}{10\sqrt{6}} = 600 - 300\sqrt{\dfrac{1}{2}} \approx$ 387.9 lbs.

13. $\dfrac{dN}{dt} = kN(200000 - N) = 200000k\,N\left(1 - \dfrac{N}{200000}\right)$ (in logistic form). We know that when $t = 0$, $N = 1$ and when $t = 1$ (week), $N = 1000$.

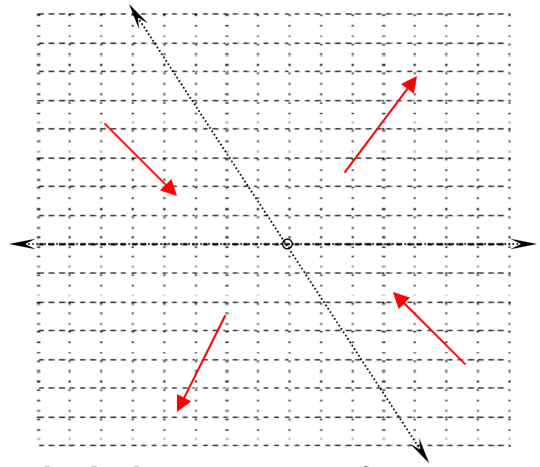
Equilibrium solutions (which are easier to find from above *before* putting it in logistic form) are obviously $N = 0$ and $N = 200000$. Since the equilibrium at $N = 200000$ is stable, the carrying capacity is 200000.

Logistic solution (without work, simply by reference) is $N(t)=\dfrac{200000}{1+\left(\dfrac{200000}{1}-1\right)e^{-200000kt}}$, and we can use our conditions to find –200000k: $1000=\dfrac{200000}{1+199999e^{-200000k}} \Rightarrow 1+199999e^{-200000k}=200$, which means $e^{-200000k}=\dfrac{199}{199999} \Rightarrow -200000k=\ln\dfrac{199}{199999}\approx$ –6.9127628. So, our final solution equation is $N(t)=\dfrac{200000}{1+199999e^{-6.9127628t}}$.

To determine how long it will take for half of the population to hear about the rumor, solve this equation with $N(t)$ = 100000. This means $100000=\dfrac{200000}{1+199999e^{-6.9127628t}}$, $1+199999e^{-6.9128t}=2$, so then $e^{-6.9128t}=\dfrac{1}{199999}$, which gives us $t=\dfrac{\ln\dfrac{1}{199999}}{-6.9128}=1.766$, so *in less than two weeks!*

14. System: $\dfrac{dx}{dt}=y$ and $\dfrac{dy}{dt}=5x+3y$. V-nullclines are found when $\dfrac{dx}{dt}=0$, so clearly y = 0 is the only one. H-nullclines are found when $\dfrac{dy}{dt}=0$, and so the only one is when 5x + 3y = 0, or $y=-\dfrac{5}{3}x$. Thus, (0, 0) is the only equilibrium point.

The plane is now divided into four regions by these nullclines. However, it's pretty convenient – we can just try the points $(\pm 1, \pm 1)$ for direction, remembering that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. So, at (1, 1) we have $\frac{dx}{dt} = 1$ (positive, so to the right), and $\frac{dy}{dt} = 8$ (positive, so upward). It is equally simple to test the other three regions. The only equilibrium point is unstable; it clearly repels nearby trajectories in two directions.



15. A tank initially contains 100 gallons of pure water. Water with a dissolved salt concentration of 1 gram per gallon begins to flow in at a rate of 2 gallons per minute. At the same time, the well-mixed salt solution in the tank is being pumped out at a rate of 3 gallons per minute. How many minutes does it take for the amount of salt in the tank to reach a maximum?

So we had established some of the basic setup for this in class the day we looked at it. We let $x(t)$ represent the grams of salt in the tank after t minutes. We know that $x(0) = 0$. We also know that the differential equation is $\frac{dx}{dt} = \text{rate in} - \text{rate out} = \left(1 \frac{\text{g}}{\text{gal}}\right)\left(2 \frac{\text{gal}}{\text{min}}\right) - \left(\frac{x}{100-t} \frac{\text{g}}{\text{gal}}\right)\left(3 \frac{\text{gal}}{\text{min}}\right) = 2 - \frac{3x}{100-t}$.

This is linear, first-order, and we can solve it using integrating factor. We have $x' + \frac{3}{100-t}x = 2$, so our integrating factor is $\mu(t) = e^{\int \frac{3}{100-t}dt} = e^{-3\ln|100-t|} = \frac{1}{(100-t)^3}$. We multiply both sides by this, which yields $\frac{1}{(100-t)^3}\left(x' + \frac{3}{100-t}x\right) = \frac{2}{(100-t)^3} \Rightarrow \int d\left(\frac{1}{(100-t)^3}x\right) = \int \frac{2}{(100-t)^3}dt$, so after integrating, we get $\frac{1}{(100-t)^3}x = \frac{1}{(100-t)^2} + C$. Since we know $x(0) = 0$, we quickly determine that $C = -\frac{1}{10000}$, so our equation is $x(t) = 100 - t - \frac{(100-t)^3}{10000}$. Now, we wish to maximize this function (essentially a Calculus problem). So, let's take its derivative and set it equal to zero… $x'(t) = -1 + \frac{3(100-t)^2}{10000} = 0$, so $3(100-t)^2 = 10000 \Rightarrow (100-t)^2 = \frac{10000}{3} \Rightarrow 100 - t = \frac{100}{\pm\sqrt{3}} \Rightarrow t = 100 \pm \frac{100}{\sqrt{3}}$. We notice that t can not be greater than 100, since the tank will be empty in 100 minutes, so $t = 100 - \frac{100}{\sqrt{3}} \approx 42.3$ minutes. The amount of salt in the tank will be maximum after 42.3 minutes. *I could ask for this to the nearest second, obviously.*