**MA 232 – Straight – Spring 2008**
**Solutions to Sample Test 2 Problems**

1. For the vector $\vec{\mathbf{x}} = [2, -1, 5, 0]$, $\|\vec{\mathbf{x}}\| = \sqrt{2^2 + (-1)^2 + 5^2 + 0^2} = \sqrt{30}$.

2. $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -3 & 0 & 2 & 0 \\ 1 & 4 & -1 & 2 \end{bmatrix}$ *can* be multiplied, since **A** is 3x2 and **B** is 2x4, so their dimensions are compatible. The product **AB** is therefore 3x4: $\begin{bmatrix} -7 & -4 & 5 & -2 \\ -9 & 0 & 6 & 0 \\ -4 & -4 & 3 & -2 \end{bmatrix}$.

3. $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, so $\mathbf{A}^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ and $\mathbf{B}^T = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Further, $(\mathbf{A} + \mathbf{B})^T = \begin{bmatrix} 8 & 10 \\ 12 & 14 \end{bmatrix}^T$, which is $\begin{bmatrix} 8 & 12 \\ 10 & 14 \end{bmatrix}$. Now, $\mathbf{A}^T + \mathbf{B}^T = \begin{bmatrix} 8 & 12 \\ 10 & 14 \end{bmatrix}$ as well, so we have verified the desired result. *Note that this is <u>not</u> a proof!*

4. The system in matrix form: $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$. As an augmented matrix, the system would appear as follows: $\left[\begin{array}{ccc|c} 1 & -3 & 2 & 4 \\ 2 & 1 & -3 & 1 \\ 1 & -1 & 1 & -2 \end{array}\right]$. *This system could easily be solved without a lot of extra time by using the matrix features of your calculator – this could be expected on the exam.*

5. The inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & 1 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -1 \end{bmatrix}$. This can be found easily on a calculator. *You could be expected to verify it is the inverse by performing the product of* **A** *and its inverse to get the identity matrix* $\mathbf{I}_3$. *You could also be asked specifically to find an inverse by using Gauss-Jordan reduction, for which you'd have to show all ERO steps.*

6. For the matrix $\mathbf{A} = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$, the determinant is simply (3)(9) – (5)(7) = 27 – 35 = –8.

7. In matrix form, the equation is $\begin{bmatrix} 2 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, or $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$. Cramer's rule says we need to find the determinant of **A**, which is $|\mathbf{A}| = 4 - (-15) = 19$. We also need two other matrices $\mathbf{A}_1$ and $\mathbf{A}_2$, which are

found by replacing a column of $\mathbf{A}$ with the values from vector $\vec{\mathbf{b}}$. So, $\mathbf{A}_1 = \begin{bmatrix} 4 & -5 \\ -2 & 2 \end{bmatrix}$, and $|\mathbf{A}_1| = -2$.
Also, $\mathbf{A}_2 = \begin{bmatrix} 2 & 4 \\ 3 & -2 \end{bmatrix}$, and $|\mathbf{A}_2| = -16$. So, the solutions are $x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = -\frac{2}{19}$ and $y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{16}{19}$.

8. Augment the given matrix with the identity: $\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$, then use Gauss-Jordan reduction to obtain RREF. This is made a littler simpler by the fact the we start with an upper triangular matrix. I'll start by negating the third row $(R_3{*} = -R_3)$: $\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$. Then, I'll do $R_2{*} = R_2 + R_3$ to get $\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$. Now, I'll do $R_1{*} = R_1 + 2R_2 - 2R_3$ (combining a couple of steps), which gives me $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$. So the inverse $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

9. Ah – a third-order DE – but it's homogeneous! So we find the characteristic equation, which is $r^3 - 5r^2 + 17r - 13 = 0$. We don't have any really good ways to solve cubics, so it better be something we can work with pretty easily. I *notice* (because I check it out) that $r$ = 1 gives me 0, so $r$ = 1 is a characteristic root. So, I can divide out the polynomial, or use synthetic division, to determine the rest.

$$\begin{array}{r|l} & r^2 - 4r + 13 \\ \hline r-1 & r^3 - 5r^2 + 17r - 13 \\ & -(r^3 - r^2) \\ & -4r^2 + 17r \\ & -(-4r^2 + 4r) \\ & 13r - 13 \\ & 13r - 13 \end{array}$$

$$\begin{array}{c|cccc} 1 & 1 & -5 & 17 & -13 \\ \hline & 1 & -4 & 13 & \mid\ 0 \end{array}$$

Ultimately, we can see that $r^3 - 5r^2 + 17r - 13 = (r-1)(r^2 - 4r + 13)$, so the characteristic roots are $r$ = 1, and $r = 2 \pm 3i$, so $\alpha = 2$ and $\beta = 3$. Thus, my general solution is $y(t) = c_1 e^t + c_2 e^{2t} \cos 3t + c_3 e^{2t} \sin 3t$.

10. 7-lb object, so mass $m$ = $\frac{7}{32}$ slugs. No friction, so damping constant $b$ = 0. To find the spring constant, we know $k = \frac{7 \text{ lb}}{4 \text{ in}} = \frac{7 \text{ lb}}{\frac{1}{3} \text{ ft}} = 21 \frac{\text{lb}}{\text{ft}}$. So, our equation is $\frac{7}{32}\ddot{x} + 21x = 0$, since no forcing function is given. Also, our initial conditions are specified: $x(0) = \frac{1}{4}$ (3 inches converted to ft), and $\dot{x}(0) = 1.5$.

To solve undamped, unforced oscillation, we calculate $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{21}{7/32}} = \sqrt{96} = 4\sqrt{6}$. Thus, our general solution is $x(t) = c_1 \cos 4\sqrt{6}t + c_2 \sin 4\sqrt{6}t$. So, $\dot{x}(t) = -c_1 4\sqrt{6} \sin 4\sqrt{6}t + c_2 4\sqrt{6} \cos 4\sqrt{6}t$ and we can find the parameters using the ICs. $\frac{1}{4} = c_1$ from the first IC, and from the second IC, we obtain $\frac{3}{2} = c_2 4\sqrt{6}$, so $c_2 = \frac{3}{8\sqrt{6}} = \frac{\sqrt{6}}{16}$. So my equation of motion is $x(t) = \frac{1}{4}\cos 4\sqrt{6}t + \frac{\sqrt{6}}{16}\sin 4\sqrt{6}t$.

11. This is kind of like problem 9, only backwards. We're given two characteristic roots, and we know there must be three, since it's a third-order DE. Since complex roots appear in conjugate pairs, we know that all three characteristic roots will be – 4, 2 – *i*, and 2 + *i*. Using the two complex roots, we can quickly write a quadratic equation using the sum and product properties: for the general quadratic equation $ar^2 + br + c = 0$, the sum of the roots is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$. Since the sum of 2 – *i* and 2 + *i* is 4, we know $-\frac{b}{a} = 4$. The product of 2 – *i* and 2 + *i* is $4 - i^2 = 5$, so $\frac{c}{a} = 5$. We can say that *a* = 1, *b* = – 4, and *c* = 5, so a quadratic equation with roots of 2 – *i* and 2 + *i* is simply $r^2 - 4r + 5 = 0$. Now, since the other characteristic root is – 4, we can just multiply this quadratic by *r* + 4, which will yield $(r+4)(r^2 - 4r + 5) = r^3 - 11r + 20 = 0$. Thus, our original DE could have been the equation $y''' - 11y' + 20y = 0$. The solution to the DE (which could have been written at the start), is simply $y(t) = c_1 e^{-4t} + c_2 e^{2t}\cos t + c_3 e^{2t}\sin t$.

12. First, put DE in standard form, since some methods (Variation of Parameters, in particular) will require it: $y'' + 2y' - 3y = \frac{3}{2}\cos t + \frac{1}{2}t^2 - \frac{5}{2}t$. (My problem originally had a typo.)

Now, find $y_h$ by solving $y'' + 2y' - 3y = 0$. Use characteristic roots: $r^2 + 2r - 3 = 0 \Rightarrow (r+3)(r-1) = 0$, so $r = -3, 1$. Thus, $y_h = c_1 e^{-3t} + c_2 e^t$.

Next, find $y_p$ (I'll do this in two pieces, $y_{p_1}$ and $y_{p_2}$.)

$y_{p_1}$: I'm solving $y'' + 2y' - 3y = \frac{3}{2}\cos t$ by undetermined coefficients, so let $y_{p_1} = A\cos t + B\sin t$. This means $y_{p_1}' = -A\sin t + B\cos t$ and $y_{p_1}'' = -A\cos t - B\sin t$. Subbing these into our DE in the previous line gives (eventually) $(-A + 2B - 3A)\cos t + (-B - 2A - 3B)\sin t = \frac{3}{2}\cos t$, so $-4A + 2B = \frac{3}{2}$ and $-2A - 4B = 0$, so $-10A = 3 \Rightarrow A = -\frac{3}{10}$, thus $B = \frac{3}{20}$. So, $y_{p_1} = -\frac{3}{10}\cos t + \frac{3}{20}\sin t$.

I now find $y_{p_2}$ by solving $y'' + 2y' - 3y = \frac{1}{2}t^2 - \frac{5}{2}t$, again using undetermined coefficients. This time, I'll set $y_{p_2} = Ct^2 + Dt + E$ (since it's a poor habit to repeat use of unknowns within a problem). That means

$y_{p_2}' = 2Ct + D$ and $y_{p_2}'' = 2C$. So (eventually) $(-3C)t^2 + (4C - 3D)t + (2C + 2D - 3E) = \frac{1}{2}t^2 - \frac{5}{2}t$, meaning that $C = -\frac{1}{6}$, $D = \frac{11}{18}$, and $E = \frac{8}{27}$. We now have $y_{p_2} = -\frac{1}{6}t^2 + \frac{11}{18}t + \frac{8}{27}$.

So, $y_p = -\frac{3}{10}\cos t + \frac{3}{20}\sin t - \frac{1}{6}t^2 + \frac{11}{18}t + \frac{8}{27}$, and I'll cop out ☺ by saying $y(t) = y_h + y_p$ is the final solution (you'll need to write it out on the exam).

13. We still need to find $y_h$ first, to determine our basis $\{y_1, y_2\}$. So, let's solve $y'' + 2y' + y = 0$. Characteristic equation is $r^2 + 2r + 1 = 0 \Rightarrow r = -1$ is the (repeated) characteristic root. The general $y_h$ solution is $y_h = c_1 e^{-t} + c_2 t e^{-t}$, and our basis functions could be $y_1 = e^{-t}$ and $y_2 = te^{-t}$. (We will also need to determine that $y_1' = -e^{-t}$ and $y_2' = -te^{-t} + e^{-t}$, or $y_2' = e^{-t}(1-t)$.)

Now, on to variation of parameters. We want to look for a particular solution $y_p = v_1 y_1 + v_2 y_2$, subject to the auxiliary condition that $y_1 v_1' + y_2 v_2' = 0$ (the first equation in the resulting system). The second equation in the resulting system comes from the general work we did in class – which does <u>not</u> need to be shown, and is $y_1' v_1' + y_2' v_2' = f(t)$. From these (general) equations, we now write our (specific) system: $e^{-t} v_1' + te^{-t} v_2' = 0$ and (from the derivatives found above) $-e^{-t} v_1' + e^{-t}(1-t) v_2' = e^{-t} \ln t$. This system is pretty easy to solve algebraically (so I won't use the Cramer's rule formulas, involving the Wronskian). I can add the two equations as they are written and get $e^{-t}(v_2') = e^{-t}\ln t$, so $v_2' = \ln t$. Then I can sub this back in (to either equation) and get $v_1' = -t \ln t$. Now I have to integrate to find $v_1$ and $v_2$.
$v_1 = \int -t \ln t \, dt = -\frac{t^2}{2}\ln t - \frac{t^2}{4}$, using integration by parts. $v_2 = \int \ln t \, dt = t \ln t - t$, also by parts. Thus, my particular solution is $y_p = e^{-t}\left(-\frac{t^2}{2}\ln t - \frac{t^2}{4}\right) + te^{-t}(t \ln t - t)$.

Final answer to the original DE: $y(t) = c_1 e^{-t} + c_2 t e^{-t} + e^{-t}\left(-\frac{t^2}{2}\ln t - \frac{t^2}{4}\right) + te^{-t}(t \ln t - t)$. Very cool…

14. Reduction of order requires that I set $y_2 = v y_1$, where $v$ is a function of $t$ (not constant). In our case, since $y_1 = t$ (it actually is a solution – I couldn't go on without checking it), we have $y_2 = vt$. So, $y_2' = v + v't$ by product rule, and $y_2'' = v' + v' + v''t = 2v' + v''t$. Let's substitute!
$t^2(2v' + v''t) - t(v + v't) + tv = 0$, which becomes $2t^2 v' + t^3 v'' - tv - t^2 v' + tv = 0$, or $t^3 v'' + t^2 v' = 0$. Now, if we let $w = v'$, then $w' = v''$, so our equation becomes $t^3 w' + t^2 w = 0$. (*This is where the <u>reduction of order</u> happens – it was a second-order DE, <u>now</u> it's a first-order DE.*) This equation can be simplified to $tw' + w = 0$ and can be solved by separation of variables.
$t\frac{dw}{dt} + w = 0 \Rightarrow t\frac{dw}{dt} = -w \Rightarrow \frac{dw}{w} = -\frac{1}{t}dt \Rightarrow \int \frac{dw}{w} = -\int \frac{dt}{t} \Rightarrow \ln|w| = -\ln|t| \Rightarrow w = \frac{1}{t}$. Since $w = v'$, we can integrate this to find $v$. $v = \int \frac{1}{t} dt = \ln|t|$. Thus, our second (linearly independent) solution is $y_2 = t \ln t$ (assuming $t$ is non-negative and dropping absolute value). So, the general solution to the original DE is $y(t) = c_1 t + c_2 t \ln t$, although this was not requested on this problem.

15. Finding eigenvalues (and corresponding eigenvectors)… Our matrix is $\mathbf{A}=\begin{bmatrix}2 & -1\\ -6 & 1\end{bmatrix}$. So the matrix $\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}2-\lambda & -1\\ -6 & 1-\lambda\end{bmatrix}$. We take the determinant of this and set it equal to zero: $(2-\lambda)(1-\lambda)-6=0$, or $\lambda^2-3\lambda-4=0$, so $(\lambda-4)(\lambda+1)=0 \Rightarrow \lambda=-1,4$. These are the eigenvalues.

To find an eigenvector (for each eigenvalue), we solve the equation $(\mathbf{A}-\lambda_i\mathbf{I})\vec{\mathbf{v}}_i=\vec{\mathbf{0}}$. So, for $\lambda=-1$, we have $\begin{bmatrix}3 & -1\\ -6 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$, or $\left[\begin{array}{cc|c}3 & -1 & 0\\ -6 & 2 & 0\end{array}\right]$ in augmented matrix form. Putting this in RREF is quite quick, giving $\left[\begin{array}{cc|c}1 & -1/3 & 0\\ 0 & 0 & 0\end{array}\right]$, so this means $v_1-\frac{1}{3}v_2=0$, or $v_2=3v_1$. Thus, any matrix in the form $\begin{bmatrix}k\\ 3k\end{bmatrix}$ will work, and most people would pick something simple like $\begin{bmatrix}1\\ 3\end{bmatrix}$ as the eigenvector to pair with the eigenvalue – 1. (*Eigenvectors that correspond to an eigenvalue are* <u>*not unique*</u> *– there will always be an infinite number of possibilities.*)

To find the second eigenvector, one that pairs with the eigenvalue 4, I do the same thing: $\begin{bmatrix}-2 & -1\\ -6 & -3\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$, or $\left[\begin{array}{cc|c}-2 & -1 & 0\\ -6 & -3 & 0\end{array}\right]$ in augmented form. RREF is $\left[\begin{array}{cc|c}1 & 1/2 & 0\\ 0 & 0 & 0\end{array}\right]$, so $v_1+\frac{1}{2}v_2=0$, or $v_2=-2v_1$. I'd pick something like $\begin{bmatrix}1\\ -2\end{bmatrix}$, although anything in the form $\begin{bmatrix}k\\ -2k\end{bmatrix}$ would be acceptable.