# RESEARCH STATEMENT

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## 1. Introduction

One of the most beautiful and central branches of mathematics is algebraic geometry. Algebraic geometers study geometric objects, called algebraic varieties, that are defined by polynomial equations. Often these geometric shapes are not smooth, but contain singularities. Singularities look like places where the object curls to intersect itself, or is otherwise pinched, twisted, or contorted. For example, a calculus student learns that the curve $y^2 = x^3$ is differentiable everywhere except at the origin; the origin here is a kind of singularity. Many mathematicians are hard at work trying to understand singularities, and have recently learned how to make use of "jet schemes" to understand them. My work lies in this growing subject.

Jet schemes, or truncated arc spaces, are geometric objects. Just as physicists study the "phase space" of a physical system, each of whose points simultaneously tells a position and a velocity, we study jet schemes of varieties. One point of a jet scheme encodes the position, velocity, and higher derivatives of a point moving along a variety. More precisely,

**Definition.** *Let $X$ be a scheme of finite type over a field $k$. An $m$-jet of $X$ is a morphism*

$$\operatorname{Spec} k[t]/(t^{m+1}) \longrightarrow X.$$

*The set of all $m$-jets carries the structure of a scheme $J_m(X)$, called the $m$th jet scheme of $X$. The arc space of $X$ is $J_\infty(X) = \varprojlim J_m(X)$.*

Jet schemes were first studied seriously by Nash [Na] who conjectured a tight relationship between the singularities of a variety and the geometry of its arc space. His ideas in modern language can be found in a recent paper of Ishii and Kollar [IK]. These schemes have generated new interest due to their appearance in Konstevich's theory of motivic integration [Ko], with which he proved Batyrev's conjecture:

**Theorem 1.** *(Kontsevich) Birationally equivalent smooth Calabi-Yau varieties have the same Hodge numbers.*

This problem was motivated by a topological mirror symmetry problem in string theory.

Recent work of Mustaţă [Mu1] showed that the arc spaces contain information about singularities; for instance, a local complete intersection has rational singularities if and only if all its jet schemes are irreducible. Furthermore, he and many others obtained certain invariants of birational geometry, such as the log canonical threshold of a pair, via the study of jet schemes and motivic integration, see [Mu2, EM, EMY].

However, there has been little study of the explicit scheme structure of jet schemes, even in simple cases. Such a study is one of the main focuses of my proposed research. For example, I have already proved many results about the scheme structure of jet schemes for monomial varieties and also for determinantal varieties. In projects 1 and 2, future problems in this direction are proposed. In a slightly different direction, I also plan to study higher dimensional analogs of jet schemes (respectively arc spaces) called truncated wedge schemes (respectively wedge spaces) and to apply jet schemes to the computation of motivic integrals of toric varieties; see projects 3 and 4.

## 2. Project 1: Jet schemes of monomial schemes

Goward and Smith [GS] began a study of the jet schemes of monomial varieties by precisely computing the minimal primes of their jet schemes. This gives a complete understanding of the irreducible components of the reduced jet schemes for monomial ideals, but no more about the scheme structure.

**Problem 1.** *Find an explicit precise description of the primary components of the jet schemes of monomial schemes.*

This is likely to be very difficult in general, especially for non-reduced monomial schemes. Therefore I have focused on several more specific problems. To describe them, we introduce the following notation.

The $m$th jet scheme of affine $r$-space $\mathbb{A}^r$ with coordinates $x_1, \ldots, x_r$ is isomorphic to affine $r(m+1)$-space $\mathbb{A}^{r(m+1)}$ with coordinates $x_1^{(0)}, x_1^{(1)}, \ldots, x_1^{(m)}, \ldots, x_r^{(0)}, x_r^{(1)}, \ldots, x_r^{(m)}$. Here $x_i^{(j)}$ represents the $j$th derivative with respect to $x_i$ of an arc. In particular, the $m$th jet scheme of a closed subscheme $X$ of $\mathbb{A}^r$ is a closed subscheme of $\mathbb{A}^{r(m+1)}$ given by some polynomials in the $x_i^{(j)}$. We will denote the ideal of this jet scheme by $J_m(X)$.

**Problem 1a.** *Describe the scheme structure of the jet schemes in the case of a reduced hypersurface monomial scheme.*

In this case, the variety $X$ is defined by a single monomial $x_1 \cdots x_r$. Goward and Smith showed that the minimal primes of $J_m(X)$ are $P(m, t_1, \ldots, t_r) = (x_1^{(0)}, \ldots, x_1^{(t_1)}, \ldots, x_r^{(0)}, \ldots x_r^{(t_r)})$ where $-1 \leq t_i \leq m$ and $\sum t_i = m + 1 - r$. (Here, we adopt the convention that the value $t_i = -1$ means the variable $x_i$ does not appear at all.)

By observing that the generators of $J_m(X)$ form a regular sequence and using the fact that length is additive on short exact sequences, I proved the following:

**Theorem 2.** *Let $R = k[x_i^{(j)}]$, $1 \leq i \leq r$, $0 \leq j \leq m$, be a polynomial ring. Then the multiplicity of $J_m(X)$ along the minimal prime $P(m, t_1, \ldots, t_r)$ is*

$$\binom{m+1}{t_1+1, t_2+1, \ldots, t_r+1}.$$

This gives a much better understanding of the jet schemes of the monomial hypersurface. In particular, it gives a feeling for how "fat" the jet schemes are along each of their components. On the other hand, a more precise understanding of the scheme structure in this case seems more difficult. For example, one can hope to describe the generators for each of the primary components. Using the software Macaulay 2, I have been able to generate the primary ideals of some jet schemes of the simple monomial scheme $\operatorname{Spec} k[x,y]/(xy)$. In the cases that the computer data has led me to a precise conjecture, I was able to prove an exact formula for the primary components.

Another case I have studied extensively, inspired by a question of Lawrence Ein, is when the monomial scheme $X$ is defined by $(x^n)$ in the affine line. Unlike the above situation, the $m$th jet schemes in this case are all irreducible.

**Problem 1b.** *When $X = \operatorname{Spec} k[x]/(x^n)$, find a formula for the multiplicity of $J_m(X)$ along its unique irreducible component.*

Using Macaulay 2, I noticed a very intriguing pattern about the multiplicity which is illustrated in the chart below.

| | $J_0(X)$ | $J_1(X)$ | $J_2(X)$ | $J_3(X)$ | $J_4(X)$ | $J_5(X)$ | $J_6(X)$ | $J_7(X)$ | $J_8(X)$ |
|---|---|---|---|---|---|---|---|---|---|
| $n$=2 | 2 | 1 | 3 | 1 | 4 | 1 | 5 | 1 | 6 |
| $n$=3 | 3 | 2 | 1 | 6 | 3 | 1 | 10 | 4 | 1 |
| $n$=4 | 4 | 3 | 2 | 1 | 10 | 6 | 3 | 1 | 20 |

So I formulated the following conjecture:

**Conjecture.** *For $X = \operatorname{Spec} k[x]/(x^n)$, the multiplicity of $J_m(X)$ is*

$$\binom{n+r-m+nr}{r+1}$$

*where $r = \lfloor \frac{m}{n} \rfloor$, the largest integer smaller than $\frac{m}{n}$.*

I have already proven this in the cases when $m < n$, $m = nq - 1$ and $m = nq - 2$ for $q \geq 2$. The case $m = nq - 1$ is solved by realizing that after localization, $q$ generators of $J_m(X)$ actually generate the maximal ideal and therefore the multiplicity is 1. The case $m = nq - 2$ is achieved by first proving that we can express each variable $x^{(0)}, \ldots, x^{(q-1)}$ as a polynomial in $x^{(q-1)}$. Then by differentiation, we get the order of each $x^{(i)}$

with respect to $x^{(q-1)}$, and finally a degree argument yields the desired result. One possible idea to complete the proof of this conjecture is to try to induce on the modulo class of $m$ in $\mathbb{Z}/n\mathbb{Z}$.

## 3. Project 2: Jet schemes of determinantal varieties

Besides monomial schemes, I am also interested in understanding the jet schemes of determinantal varieties. Determinantal ideals are known to be Cohen-Macaulay normal domains. Mustaţă [Mu1] showed that the jet schemes of the determinantal variety defined by all $2 \times 2$ minors of a generic $2 \times n$ matrix are all irreducible.

**Problem 2.** *Analyze the scheme structure of the jet schemes of the determinantal varieties given by the vanishing of all $t \times t$ minors of a generic $r \times s$ matrix. For example, are they irreducible? If not, how many components do they have? What are their dimension? What is the multiplicity along each of the components?*

Calculations I have done with Macualay 2 have already revealed that Mustaţă's result does not hold for larger generic matrices. In fact, I have proven the following:

**Theorem 3.** *Let $X$ be the variety defined by all $2 \times 2$ minors of an $r \times s$ matrix of indeterminates. Assume $r, s \geq 3$. Then*
*(i) For $k \geq 1$ and $r+s \geq 7$, $J_{2k-1}(X)$ is reducible, with $k+1$ components of dimension $2(k-q)(r+s-1)+qrs$ where $q = 0, \ldots, k$. So $\dim(J_{2k-1}(X)) = krs$.*
*(ii) For $k \geq 0$ and $r+s \geq 7$, $J_{2k}(X)$ is reducible, with $k+1$ components of dimension $(2k-2q+1)(r+s-1)+qrs$ where $q = 0, \ldots, k$. So $\dim(J_{2k}(X)) = r+s-1+krs$.*

My method was to first observe that in the case of $2 \times 2$ minors, the singular locus of the determinantal variety is simply the origin. Then I analyzed the natural projection map $\pi_m : J_m(X) \to X$ by decomposing $J_m(X)$ into the preimage of the origin and the pullback of the smooth locus, which is easier to work with.

As mentioned earlier, Mustaţă proved that a local complete intersection has rational singularities if and only if all its jet schemes are irreducible. Note that the variety given by the vanishing of all $2 \times 2$ minors of a square generic matrix is Gorenstein and has rational singularities. Thus the above result gives examples that his theorem does not hold if we replace the local complete intersection hypothesis by Gorenstein.

The case $r = s = 3$ is more subtle. A quick calculation tells us that $\dim(\pi_m^{-1}(0)) = \dim(\overline{\pi_m^{-1}(X-0)}) - 1$. But Macaulay computations tell me that $\pi_1^{-1}(0)$ and $\pi_2^{-1}(0)$ do contribute a component of $J_1(X)$ and $J_2(X)$ respectively. Since $J_m(X) = \pi_m^{-1}(X_{reg}) \coprod \pi_m^{-1}(X_{sing})$ and $\overline{\pi_m^{-1}(X-0)}$ is irreducible here, the computer data lead me to believe that $\overline{\pi_m^{-1}(X-0)} \not\supset \pi_m^{-1}(0)$, which remains to be checked.

In order to attack the general case, note that the singular locus of the more general space of $r \times s$ matrices with vanishing $n \times n$ minors is not just the origin. However, it is another determinantal variety, defined by all the $(n-1) \times (n-1)$ minors of the matrix. I intend to analyze it by an inductive argument on $n$.

## 4. Project 3: Truncated wedge schemes

While my research began with jet schemes, my work is not confined to that topic alone. I am also interested in the study of truncated wedge schemes, which are higher dimensional analogs of jet schemes.

**Definition.** *Let $X$ be a scheme of finite type over a field $k$. A truncated $m$-wedge of $X$ is a morphism*

$$\operatorname{Spec} k[s,t]/(s,t)^{m+1} \longrightarrow X.$$

*The collection of all truncated $m$-wedges has a natural scheme structure, called the truncated $m$th wedge scheme and is denoted by $W_m(X)$.*

**Problem 3.** *Understand the components of the truncated wedge schemes of monomial schemes.*

So far, I have proven that although the truncated wedge spaces of a monomial scheme are not themselves monomial, their reduced subschemes are. This is achieved by reducing and explicitly computing the minimal primes of the truncated wedge schemes of principal monomial ideals. For hypersurfaces, this result reads:

**Theorem 4.** *Let $X$ be the scheme defined by the principal monomial ideal generated by $x_1^{a_1} \cdots x_r^{a_r}$. Then the minimal primes of the space of truncated $m$-wedges $W_m(X)$ are precisely the minimal members of the set of ideals*

$$(x_k^{(i_k,j_k)} : 0 \leq i_k + j_k \leq t_k)$$

*where* $-1 \leq t_k \leq m$ *(with the convention that* $t_k = -1$ *means that the variable* $x_k$ *doesn't appear), and* $\sum a_k(t_k+1) \geq m+1$.

Remarkably, computer calculations indicate that unlike the case of jet schemes, truncated wedge schemes may be all of multiplicity 1 along their components.

**Problem 3a.** *Prove that all the components of* $W_m(X)$ *are reduced when* $X$ *is given by the vanishing of* $x_1^{a_1}\cdots x_r^{a_r}$.

Besides doing computational algebraic geometry, I have also studied some basic properties of wedge spaces. In fact I found that quite a few of the nice properties of jet schemes and arc spaces remain true for wedge spaces. For example,

**Proposition.**

(1) *If* $X \to Y$ *is an étale morphism, then* $W_m(X) \cong W_m(Y) \times_Y X$.
(2) *If* $X$ *is a smooth, connected variety of dimension* $n$*, then for every* $m$*, the morphism* $\pi_m$ *is an affine bundle with fiber* $\mathbb{A}^{\frac{1}{2}m(m+3)n}$. *In this case,* $W_m(X)$*is smooth, connected of dimension* $\frac{1}{2}(m+1)(m+2)n$.
(3) *Let* $X$ *be locally a complete intersection of dimension* $n$ *and* $m \in \mathbb{Z}_{>0}$. *The scheme* $W_m(X)$ *is pure dimensional iff* $\dim W_m(X) = \frac{1}{2}(m+1)(m+2)n$*, and in this case* $W_m(X)$ *is locally a complete intersection. Similarly,* $W_m(X)$ *is irreducible iff* $\dim \pi_m^{-1}(X_{sing}) < \frac{1}{2}(m+1)(m+2)n$.
(4) *For any scheme* $X$*,* $W_1(X) \cong TX \times_X TX$.

## 5. Project 4: p-adic integrals and motivic integrals

Igusa zeta function of an integer polynomial $f$ is a generating function for the solutions of $f \bmod p^{m+1}$. Denef and Loeser defined in [DL] a motivic analogue, called motivic integral, by replacing $\mathbb{Z}/p^{m+1}\mathbb{Z}$ by $\mathbb{C}/(t^{m+1})$. So its solutions are $m$-jets and the computation is done in a suitable Grothendieck ring. There is an outstanding open question relating the poles of Igusa zeta function of $f$ and its Bernstein-Sato polynomial, an invariant of the singularities of $f$, which comes from $D$-module theory.

**Conjecture.** *(Igusa) Let* $f$ *be a non-constant polynomial in* $\mathbb{Z}[x_1,\ldots,x_n]$. *For almost all primes* $p$ *the following holds: if* $s$ *is a pole of* $Z_f$*, then the real part of* $s$ *is a root of the Bernstein-Sato polynomial* $b_f$ *of* $f$.

Mustaţă, Howald and I [HMY] proved Igusa's conjecture in the special case when $f$ is a monomial, and also proved an analogue when $f$ is replaced by any monomial ideal. We first showed that the real parts of the poles of the Igusa zeta function of a monomial ideal can be computed from the torus-invariant divisors in the normalized blow-up of the affine space along the ideal. Then using the explicit descriptions of the roots of the Bernstein-Sato polynomial of a monomial ideal, we obtained the final result. We think that the point of view of isolating the series corresponding to cones could be useful in other situations. For example, Mustaţă has used it to look at Igusa zeta functions for hyperplane arrangements. It would be interesting to see how this method could be applicable in studying other examples.

## 6. Long term research goals

In addition to the specific projects described in the previous sections, I would like to do further investigations into the relationship between singularities of a scheme and the geometry of its arc space. Mustaţă and others have made use of topological properties such as irreducibility and components dimension in understanding certain singularities. Hara, Smith and a few others [HW, Sm] have studied certain singularities via tight closure. Although these two different powerful approaches share many features, we do not understand the tools and methods of either approach in terms of the other. Can jet schemes be understood as an application of tight closure? Can tight closure be described in terms of arc spaces? Is there any close relationship between jet schemes and tight closure? One of my research goals is to try to address some of these questions. I plan to start by looking at some cases where both approaches are well understood so that I can compare each picture, for example, the cusp or the cones over elliptic curves.

I would also like to study what information about singularities can be obtained from wedges. Unfortunately, unlike arcs, wedges do not in general lift even via a proper birational morphism of smooth schemes. So one of my research goals is to investigate what kinds of morphism of smooth schemes and more generally schemes of finite type allow lifting of wedges. To start, I plan to investigate how much information about wedges can be obtained from looking at a resolution of singularities. In the case of arcs, everything follows from the change of variable formula. For instance, this formula is very easy to see directly for blow-ups with smooth centers. Therefore a first step would be to see how wedges behave under this transformation. One of the obstacles is attaining the change of variable formula, because a crucial fact used by Kontsevich is that $k[[t]]$ is a PID.

## References


[DL] J. Denef and F. Loeser, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), 505–537.

[EM] L. Ein and M. Mustaţă, *The log canonical threshold of homogeneous affine hypersurfaces*, arXiv: math.AG/0105113.

[EMY] L. Ein, M. Mustaţă and T. Yasuda, *Jet schemes, log discrepancies and inversion of adjunction*, Invent. Math. **153** (2003), no.3, 519–535.

[GS] R. Goward Jr. and K.E. Smith, *The jet scheme of a monomial scheme*, preprint (2005), available online: `http://www.math.lsa.umich.edu/~kesmith`.

[HMY] J. Howald, M. Mustaţă and C. Yuen, *On Igusa zeta functions of monomial ideals*, arXiv: math.AG/0509243.

[HW] N. Hara and K.-i. Watanabe, *F-regular and F-pure rings vs. log terminal and log canonical singularities*, J. Algebraic Geom. **11** (2002), no.2, 363–392.

[IK] S. Ishii and J. Kollár, *The Nash problem on arc families of singularities*, Duke Math. J. **120** (2003), no.3, 601–620.

[Ko] M. Kontsevich, *Lecture at Orsay*, 1995.

[Mu1] M. Mustaţă, *Jet schemes of locally complete intersection canonical singularities*, Invent. Math. **145** (2001), 397–424.

[Mu2] M. Mustaţă, *Singularities of pairs via jet schemes*, J. Amer. Math. Soc. **15** (2002), 599–615.

[Na] J. Nash, Jr., *Arc structure of singularities*, Duke Math. J. **81** (1995), no.1, 31–38.

[Sm] K.E. Smith, *F-rational rings have rational singularities*, Amer. J. Math. **119** (1997), no.1, 159–180.